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A NODAL SPLINE COLLOCATION METHOD FOR WEAKLY SINGULAR VOLTERRA INTEGRAL EQUATIONS

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Dedicated to Professor Gheorghe Micula at his 60^{th} *anniversary*

Abstract. A collocation method based on optimal nodal splines is presented for the numerical solution of linear Volterra integral equations of the second kind with weakly singular kernel. Since the considered spline operator is a bounded projector we can prove that, for sequences of locally uniform meshes, the approximate solution error converges to zero at exactly the same optimal rate as the spline approximation error. We consider in particular sequences of graded meshes, for which the local uniformity is proved. Finally, we give an upper bound for the condition number of the collocation system and we present some numerical examples.

1. **Introduction**

The Volterra integral equation of the second kind

$$y(x) = f(x) + \int_0^x k(x,s)y(s)ds, \quad x \in I \equiv [0,X] \tag{1}$$

with weakly singular kernel k provides mathematical model describing a wide variety of applicative problems. Particularly interesting kernels are the convolution ones, of the form $k(x-s)$, where $k(t) \in C(O,X] \cap L_1(O,X)$, but $k(t)$ may become unbounded as $t \to 0$. Examples of convolution kernels are

$$k(t) = \lambda |t|^{-\alpha} \quad , \quad 0 < \alpha < 1 \tag{2}$$

$$k(t) = \lambda \log |t| \, , \tag{3}$$

where $\lambda \in \mathbf{R}$.

If $f \in C(I)$, then (1) has a unique solution $y \in C(I)$. As f becomes smoother, y also becomes smoother, but only for $x > 0$. In general there will be no increase in smoothness of the solution at $x = 0$. At the same time, very special choices of f may force smoother behaviour at the origin [13].

In the recent literature, some collocation methods, based on piecewise polynomials for solving (1) with the above kernels, have been studied (cfr. [2,12] and

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references therein). In order to find an approximate solution sufficiently smooth in $(0, X]$, one may use polynomial splines of order m, belonging to $C^{\nu}(I), 0 \le \nu \le m-2$.

In this context we propose a new product collocation method, for numerically solving (1), based on optimal nodal splines of order $m > 2$. We generate a sequence of spline approximations $\{y_n\}$ for the solution of (1) and we analyze its convergence to y. Since the constructed approximation operator is a bounded projection operator, it will be proved that $||y - y_n||$ converges to zero at exactly the same rate as the norm of nodal spline approximation error for sequences of locally uniform (l.u.) meshes.

In order to reflect the possible singular behaviour of the solution near to the initial point, we will resort to a sequence of graded meshes. Indeed in this context we will prove also that the above sequence is l.u..

The paper is organized as follows. In Section 2 we give some preliminaries relative to the nodal spline space of our interest, the construction and convergence properties of the approximating operator. In Section 3 we give our spline collocation method for the problem (1). Section 4 is devoted to the error analysis and in Section 5 we study the condition number for the collocation method. Finally, in Section 6 we present some numerical results; in one case, in particular, we will show the better performance of the sequence of graded partitions with respect to the uniform one, when the solution has the first derivative singular at $x = 0$.

2. **On optimal nodal splines**

We briefly review the definition and the main properties of the optimal nodal splines of interest in this context [5-8].

Let $I = [0, X]$ be a given finite interval of the real line $\mathbf{R}$, for a fixed integer $m \ge 3$ and $n \ge m - 1$, we define a partition Π_n of I by

$$\Pi_n : 0 = \tau_0 < \tau_1 < ... < \tau_n = X \ ,$$

generally called "primary partition". We insert $m - 2$ distinct points throughout $(\tau_\nu, \tau_{\nu+1})$, $\nu = 0, ..., n-1$ obtaining a new partition of I

$$X_n : 0 = x_0 < x_1, < ... \qquad < x_{(m-1)n} = X,$$

where $x_{(m-1)i} = \tau_i$, $i = 0, ..., n$.

Let

$$R_n = \max_{\substack{0 \le k,j \le n-1 \\ |k-j|=1}} \frac{\tau_{k+1} - \tau_k}{\tau_{j+1} - \tau_j} \ , \tag{4}$$

we say that the sequence of primary partitions $\{\Pi_n; n = m-1, m, ...\}$ is l.u. if, for all n, there exists a constant $A \ge 1$ such that $R_n \le A$, i.e.

$$\frac{1}{A} \le \frac{\tau_{k+1} - \tau_k}{\tau_{j+1} - \tau_j} \le A \ , \ k, j = 0, 1, ..., n-1 \text{ and } |k - j| = 1 \ . \tag{5}$$

Since the convergence results of the nodal splines we shall consider are based on the local uniformity property of the primary partitions sequence and one of our objectives is the use of graded meshes, in the following proposition we shall prove that a sequence of primary graded partitions is l.u.

Proposition 1. *Let $[0, X]$ be a finite interval. The sequence of partitions $\{\Pi_n\}$, obtained by using graded meshes [3] of the form*

$$\tau_i = \left(\frac{i}{n}\right)^r X \quad , \quad 0 \le i \le n \, , \tag{6}$$

with grading exponent $r \in \mathbf{R}$ assumed ≥ 1, is l.u., i.e. it satisfies (5) with $A = 2^r - 1$.

Proof. For $r = 1$, the partition is uniform and (5) is satisfied with $A = 1$.

Consider now $r > 1$ and $k = j + 1$. We can write

$$f(j) = \frac{\tau_{j+2} - \tau_{j+1}}{\tau_{j+1} - \tau_j} = \frac{\left(1 + \frac{1}{j+1}\right)^r - 1}{1 - \left(1 - \frac{1}{j+1}\right)^r} \quad , \quad j = 0, 1, ..., n-2$$

and $f(0) = 2^r - 1$.

Consider now the function $f(x) = \frac{\left(1+\frac{1}{x+1}\right)^r - 1}{1 - \left(1 - \frac{1}{x+1}\right)^r}$, $x \in \mathbf{R}^+$. Then $f(j) = f(x), x \in N$. One can verify that $\lim_{x\to\infty} f(x) = 1$ and $f'(x) < 0$ for all x.

Then

$$1 \le f(j) \le 2^r - 1 \, . \tag{7}$$

If $k = j - 1$, for $j = 1, 2, ..., n - 1$ we have

$$\frac{\tau_j - \tau_{j-1}}{\tau_{j+1} - \tau_j} = \frac{1}{\frac{(j+1)^r - j^r}{j^r - (j-1)^r}} = \frac{1}{f(j-1)}$$

and using (7), the thesis follows. ∎

Now, after introducing two integers [5]

$$i_0 = \begin{cases} \frac{1}{2}(m+1) & m \quad \text{odd} \\ \\ \frac{1}{2}m + 1 & m \quad \text{even} \end{cases} \qquad \text{and} \quad i_1 = (m+1) - i_0$$

and two integer functions

$$p_\nu = \begin{cases} 0 & \nu = 0, 1, ..., i_1 - 2 \\ \nu - i_1 + 1 & \nu = i_1 - 1, ..., n - i_0 \\ n - (m-1) & \nu = n - i_0 + 1, ..., n-1 \end{cases}$$

$$q_\nu = \begin{cases} m - 1 & \nu = 0, 1, ..., i_1 - 2 \\ \nu + i_0 & \nu = i_1 - 1, ..., n - i_0 \\ n & \nu = n - i_0 + 1, ..., n-1 \end{cases}$$

consider the set $\{w_i(x); i = 0, 1, ..., n\}$ of functions defined as follows [6,7]

$$w_i(x) = \begin{cases} l_i(x) & x \in [\tau_0, \tau_{i_1 - 1}], & i \le m - 1 \\ s_i(x) & x \in (\tau_{i_1 - 1}, \tau_{n - i_0 + 1}), & n \ge m \\ \bar{l}_i(x) & x \in [\tau_{n - i_0 + 1}, \tau_n], & i \ge n - (m-1) \end{cases}$$

where

$$l_i(x) = \prod_{\substack{k=0 \\ k\neq i}}^{m-1} \frac{x-\tau_k}{\tau_i-\tau_k} \tag{8}$$

$$\bar{l}_i(x) = \prod_{\substack{k=0 \\ k\neq n-i}}^{m-1} \frac{x-\tau_{n-k}}{\tau_i-\tau_{n-k}} \tag{9}$$

$$s_i(x) = \sum_{r=0}^{m-2} \sum_{j=j_0}^{j_1} \alpha_{i,r,j} B_{(m-1)(i+j)+r}(x) \tag{10}$$

with $j_0 = \max\{-i_0, i_1-2-i\}$, $j_1 = \min\{-i_0+m-1, n-i_0-i\}$. The coefficients $\alpha_{i,r,j}$ are given in [5] and the B-splines sequence is constructed from the set of the normalized B-splines defined in [14] for $i = (m-1)(i_1-2), (m-1)(i_1-2)+1, ..., (m-1)(n-i_0+1)$. Then, the following locality property holds [6]

$$s_i(x) = 0 \quad , \quad x \notin [\tau_{i-i_0}, \tau_{i+i_1}]. \tag{11}$$

Each $w_i(x)$ is nodal with respect to Π_n, in the sense that

$$w_i(\tau_j) = \delta_{i,j} \quad , \quad i,j = 0,1,...,n\,. \tag{12}$$

Therefore, being $\det[w_i(\tau_j)] \neq 0$, the functions $w_i(x), i = 0,1,...,n$, are linearly independent. Let $S_{\Pi_n} = \mathrm{span}\{w_i(x); i = 0,1,...,n\}$, it is proved in [7] that, for all $s \in S_{\Pi_n}$, one has $s \in C^{m-2}(I)$.

For all $g \in B(I)$, where $B(I)$ is the set of real-valued functions on I, we consider the spline operator $W_n : B(I) \to S_{\Pi_n}$, so defined

$$W_n g = \sum_{i=0}^{n} g(\tau_i) w_i(x) \quad , \quad x \in I\,.$$

By (12), for $0 \leq \nu < n$ we can write:

$$W_n g = \sum_{i=p_\nu}^{q_\nu} g(\tau_i) w_i(x), \quad x \in [\tau_\nu, \tau_{\nu+1}]\,. \tag{13}$$

It is proved in [6,7] that $W_n p = p$, for all $p \in \mathbf{P}_m$, where $\mathbf{P}_m$ denotes the set of polynomials of order m (degree $\leq m-1$), and $W_n g(\tau_i) = g(\tau_i)$, for $i = 0,1,...,n$, i.e. W_n is an interpolatory operator.

Using the results in [6,7,8] we deduce that, for l.u. $\{\Pi_n\}$, W_n is a bounded projection operator in S_{Π_n}. In fact, it is easy to show that

$$W_n s = s \quad , \quad \text{for all } s \in S_{\Pi_n}$$

and,if we denote:

$$||W_n|| = \sup\{||W_n h|| : h \in C(I), ||h|| < 1\},$$

with $||h|| = \max_{x \in I} |h(x)|$, considering that

$$||W_n|| \leq (m+1) \left[\sum_{\lambda=1}^{m-1} (R_n)^\lambda \right]^{m-1} ,$$

where R_n is defined in (4), from (5) we obtain $||W_n|| < \infty$.

Finally, for all $g \in C^\nu(I)$, $0 \leq \nu < m$, assuming that $\{\Pi_n\}$ is l.u., there results

$$||g - W_n g|| = O(H_n^\nu \omega(g^{(\nu)}; H_n; I)), \tag{14}$$

where $H_n = \max_{1 \leq i \leq n} (\tau_i - \tau_{i-1})$ and, for all $g \in C(I)$, $\omega(g; \delta; I) = \max_{\substack{x, x+h \in I \\ 0 < h \leq \delta}} |g(x+h) - g(x)|$.

3. **Spline collocation method**

Consider now the linear integral equation (1) and a sequence of nodal spline spaces $\{S_{\Pi_n}; n = m-1, m, ...\}$ spanned by $\{w_i(x); i = 0, ..., n\}$ and based on a sequence of l.u. primary partitions $\{\Pi_n\}$.

For some fixed n we consider a spline $y_n \in S_{\Pi_n}$ written in the form

$$y_n(x) = \sum_{j=0}^{n} \alpha_j w_j(x), \alpha_j \in \mathbf{R} \, . \tag{15}$$

Substituting (16) in (1) we obtain

$$y_n(x) - \int_0^x k(x, s) y_n(s) ds + r_n(x) = f(x) \, ,$$

where $r_n(x)$ is the residual term obtained approximating y by y_n in (1).

The values α_j in (16), with $j = 0, 1, ..., n$, are choosen by requiring that

$$r_n(\tau_j) = 0, \quad j = 0, 1, ..., n \, . \tag{16}$$

This leads to determine $\alpha_0, \alpha_1, ..., \alpha_n$ as the solution of a linear system that, using (13), can be written in the form:

$$\alpha_j [1 - \mu_j(\tau_j)] - \sum_{\substack{i=0 \\ i \neq j}}^{n} \mu_i(\tau_j) \alpha_i = f(\tau_j), \quad j = 0, 1, ..., n \, , \tag{17}$$

where

$$\mu_i(\tau_j) = \int_0^{\tau_j} k(\tau_j, s) w_i(s) ds \, . \tag{18}$$

By (8)-(12) and (14) we can explicitly write each weight of the set $\{\mu_i(\tau_j); i, j = 0, 1, ..., n\}$ as follows.

For $i = 0, 1, ..., m-1$:

$$\mu_i(\tau_j) = \begin{cases} 0 & j = 0 \\ \int_{\tau_0}^{\tau_j} k(\tau_j, s) l_i(s) ds & 0 < j \le i_1 - 1 \\ \int_{\tau_0}^{\tau_{i_1-1}} k(\tau_j, s) l_i(s) ds + \int_{\tau_{i_1-1}}^{\tau_j} k(\tau_j, s) s_i(s) ds \,, \\ i_1 - 1 < j \le i + i_1 \\ \int_{\tau_0}^{\tau_{i_1-1}} k(\tau_j, s) l_i(s) ds + \int_{\tau_{i_1-1}}^{\tau_{i_1+i}} k(\tau_j, s) s_i(s) ds \,, \; i + i_1 < j \le n \,. \end{cases}$$

For $i = m, ..., n-m$:

$$\mu_i(\tau_j) = \begin{cases} 0 & 0 \le j \le i - i_0 \\ \int_{\tau_{i-i_0}}^{\tau_j} k(\tau_j, s) s_i(s) ds & i - i_0 < j \le i + i_1 \\ \int_{\tau_{i-i_0}}^{\tau_{i+i_1}} k(\tau_j, s) s_i(s) ds & i + i_1 < j \le n \,. \end{cases}$$

For $i = n-m+1, ..., n$:

$$\mu_i(\tau_j) = \begin{cases} 0 \quad 0 \le j \le i - i_0 \\ \int_{\tau_{i-i_0}}^{\tau_j} k(\tau_j, s) s_i(s) ds \qquad i - i_0 < j \le n - m + i_1 \\ \int_{\tau_{i-i_0}}^{\tau_{n-m+i_1}} k(\tau_j, s) s_i(s) ds + \int_{\tau_{n-m+i_1}}^{\tau_j} k(\tau_j, s) \bar{l}_i(s) ds, \\ n - m + i_1 < j \le n \,. \end{cases}$$

We remark that writing the system (17) in the form $A\underline{\alpha} = \underline{f}$, where $A = \{a_{ji}\}_{j,i=0}^{n}$ is the coefficient matrix, $\underline{\alpha} = [\alpha_0 ... \alpha_n]^T$, $\underline{f} = [f(\tau_0) ... f(\tau_n)]^T$, the entries of A are as follows:

$$a_{jj} = \begin{cases} 1 & j = 0 \\ 1 - \mu_j(\tau_j) & j = 1, ..., n \end{cases} \tag{19}$$

and for $j \ne i$:

$$a_{ji} = \begin{cases} -\mu_i(\tau_j) & i = 0, ..., m-1; \qquad j = 1, ..., n \\ & i = m, ..., n; \qquad j = i - i_0 + 1, ..., n \\ \\ 0 & i = 1, ..., m-1; \qquad j = 0 \\ & i = m, ..., n; \qquad j = 0, ..., i - i_0 \,. \end{cases} \tag{20}$$

The algorithm for the numerical evaluation of $\{\mu_i(\tau_j)\}$ is based on the procedure given in [4] and on the knowledge of integrals of the type

$$\int_{x_r}^{x_{r+1}} k(\tau_j, s) s^\nu ds \quad , \quad \nu = 0, 1, ..., m-1 \,. \tag{21}$$

For some kernels, as those ones given in (2) and (3), the integrals (22) can be easily evaluated in a closed form [10].

Once we have the solution $\underline{\alpha}$ of the system (18), by (16) we can obtain the approximation $y_n(x)$ of the solution $y(x)$ of (1).

4. **Error analysis**

In order to carry out the error analysis for the proposed method, we write the integral equation (1) in the operator form

$$(I - \widetilde{K})y = f \,, \tag{22}$$

where

$$\widetilde{K}y = \int_I \widetilde{k}(x, s) y(s) ds, \quad x \in I \tag{23}$$

and

$$\widetilde{k}(x, s) = \begin{cases} k(x, s) \quad , & 0 \le s \le x \\ 0, & s > x \end{cases} \tag{24}$$

We remark that, for the kernels $k(x, s)$ considered in Section 1, $\widetilde{k}(x, s)$ satisfies the following properties:

(i) $\widetilde{k}(x, s)$ is Riemann – integrable as a function of s , for all $x \in I$,

(ii) $\lim\limits_{x \to x'} \int_I |\widetilde{k}(x', s) - \widetilde{k}(x, s)| ds = 0$, for $x', x \in I$,

(ii) $\max\limits_{x \in I} \int_I |\widetilde{k}(x, s)| ds < \infty$.

Therefore, we conclude that the operator $\widetilde{K}$ is a bounded compact operator on $C(I)$.

In Section 2 it has been remarked that, considering a sequence of l.u. primary partitions $\{\Pi_n\}$, the spline operator W_n is a bounded interpolating projection operator, then the condition (17) can be rewritten as

$$W_n r_n = 0 \quad \text{or, equivalently,}$$

$$(I - W_n \widetilde{K}) y_n = W_n f \,. \tag{25}$$

Now, we will prove that the equation (26) has a unique solution y_n. Then we will study the convergence of y_n to y and we will give an upper bound for $||y - y_n||$.

In order to get the above results, we prove the following lemma.

Lemma 1. *Given a sequence of l.u. partitions* $\{\Pi_n\}$*, for the sequence of projections* $\{W_n : C(I) \to S_{\Pi_n}\}$*, there results*

$$||\widetilde{K} - W_n\widetilde{K}|| \to 0 \text{ as } n \to \infty \tag{26}$$

Proof. Being $\widetilde{K} : C(I) \to C(I)$ a compact operator and since (15) with $\nu = 0$ holds, we obtain the convergence result (27). ∎

Theorem 1. *Let* $\{\Pi_n\}$ *be a sequence of l.u. partitions. Consider the bounded projection operator* W_n *from* $C(I)$ *to* S_{Π_n}.

For all n *sufficiently large, say* $n \geq N$*, the operator* $(I - W_n\widetilde{K})^{-1}$ *from* $C(I)$ *to* $C(I)$ *exists. Moreover it is uniformly bounded, i.e.:*

$$\sup_{n\geq N} ||(I - W_n\widetilde{K})^{-1}|| \leq M < \infty \tag{27}$$

and

$$||y - y_n|| \leq ||(I - W_n\widetilde{K})^{-1}||\, ||y - W_n y|| \,. \tag{28}$$

This leads to $||y - y_n||$ converging to zero exactly with the same rate of $||y - W_n y||$.

Proof. Adapting properly the results in [1], we write:

$$I - W_n\widetilde{K} = (I - \widetilde{K})[I - (I - \widetilde{K})^{-1}(W_n\widetilde{K} - \widetilde{K})] \,.$$

Using Lemma 1, we can find an integer N such that

$$\varepsilon_N = \sup_{n\geq N} ||\widetilde{K} - W_n\widetilde{K}|| < \frac{1}{||(I - \widetilde{K})^{-1}||} \,.$$

Then, for $n \geq N$, the inverse of $[I - (I - \widetilde{K})^{-1}(W_n\widetilde{K} - \widetilde{K})]$ exists and exploiting the geometric series theorem, there results

$$||[I - (I - \widetilde{K})^{-1}(W_n\widetilde{K} - \widetilde{K})]^{-1}|| \leq \frac{1}{1 - \varepsilon_N ||(I - \widetilde{K})^{-1}||} \,.$$

Therefore:

$$||(I - W_n\widetilde{K})^{-1}|| \leq \frac{||(I - \widetilde{K})^{-1}||}{1 - \varepsilon_N ||(I - \widetilde{K})^{-1}||} \equiv M < \infty \,. \tag{29}$$

In order to show (29) we multiply (23) by W_n and then rearrange to obtain

$$(I - W_n\widetilde{K})y = W_n f + (I - W_n)y \tag{30}$$

If we subtract (26) from (31) we obtain

$$y - y_n = (I - W_n\widetilde{K})^{-1}(y - W_n y) \,,$$

and using (30) the thesis follows. ∎

5. Condition number of the collocation method

We can also obtain an upper bound for the condition number of the linear system (18), by adapting some general results in [1].

For a given matrix $B \in \mathbf{R}^{d\times d}$ we will use the row norm so defined:

$$||B|| = \max_{0\le j\le (d-1)} \sum_{i=0}^{d-1} |B_{j,i}| \,.$$

If we denote by $\Gamma_n = [w_i(\tau_j)]_{i,j=0}^n$, using (13), there results $\Gamma_n = I$. Thus we can write

$$||A^{-1}|| \le ||W_n||\, ||\Gamma_n^{-1}||\, ||(I - W_n\widetilde{K})^{-1}|| = ||W_n||\, ||(I - W_n\widetilde{K})^{-1}|| \,.$$

From (20), (21) we obtain:

$$\sum_{i=0}^{n} |a_{j,i}| \le \sum_{i=0}^{n} |\mu_i(\tau_j)| + 1 \,.$$

Therefore, setting $||\widetilde{K}|| = \max_{0\le t\le X} \int_0^X |\widetilde{k}(t,s)|ds$, there results:

$$||A|| \le \max_{0\le j\le n} \sum_{i=0}^{n} \int_0^{\tau_j} |k(\tau_j, s) w_i(s)|ds + 1 \le ||W_n||\, ||\widetilde{K}|| + 1$$

and then

$$\mathrm{cond}(A) \le ||W_n||\, ||(I - W_n\widetilde{K})^{-1}||\, (||W_n||\, ||\widetilde{K}|| + 1) \,.$$

6. Numerical examples

In order to test the proposed method, we consider equations of the type (1) with

$$k(x,s) = \lambda(x-s)^{-\frac{1}{2}}, \quad x \in [0,1], \quad \lambda \in \mathbf{R}$$

In particular, we shall present some numerical results in the following cases:

$$\lambda = -\frac{1}{4}, \quad f(x) = \frac{1}{\sqrt{1+x}} + \frac{\pi}{8} - \frac{1}{4}\sin^{-1}\frac{1-x}{1+x} \,, \tag{31}$$

for which the exact solution is $\frac{1}{\sqrt{1+x}}$ and

$$\lambda = -1 \quad , \quad f(x) = \sqrt{x} + \frac{1}{2}\pi x \,, \tag{32}$$

for which the exact solution is $y(x) = \sqrt{x}$.

Referring to the equation defined by (32) we use our collocation method, based on cubic nodal splines ($m = 4$) with uniform primary partition Π_n, for increasing values of n. We report in Table 1 the corresponding absolute errors $|y(x) - y_n(x)|$ evaluated at the coinciding collocation points. In the last row of the table we also present the collocation matrix condition numbers.

Table 1

$|y(x) - y_n(x)|$ *for the equation (32)*, $m = 4$

x	$n = 10$	$n = 20$	$n = 40$
.1	0.10E-5	0.37E-7	0.15E-8
.2	0.55E-6	0.24E-7	0.94E-9
.3	0.39E-6	0.18E-7	0.12E-8
.4	0.30E-6	0.14E-7	0.47E-9
.5	0.24E-6	0.12E-7	0.76E-9
.6	0.21E-6	0.10E-7	0.11E-8
.7	0.18E-6	0.93E-8	0.11E-8
.8	0.15E-6	0.78E-8	0.57E-9
.9	0.58E-7	0.10E-7	0.37E-8
1.0	0.21E-6	0.80E-8	0.95E-9
condition number	1.35	1.35	1.34

Now we consider the equation defined by (33), whose exact solution $y(x) = \sqrt{x}$ has unbounded derivatives at $x = 0$. We use our collocation method and we remark that the knowledge of the behaviour of the solution suggests the use of a sequence of graded primary meshes of the form (6). Indeed we have proved in Section 2 that such a sequence of partitions Π_n is l.u., ensuring that the hypotheses of Theorem 1 are satisfied.

In Table 2, for increasing values on n, we compare absolute errors $|y(x) - y_n(x)|$, obtained using quadratic nodal splines and uniform partitions, with those ones resulting with the same splines and graded meshes of the form (6), with $r = 2$. As it was expected, the choice of graded primary partitions allows to obtain more accurate results in particular in a neighbouring of $x = 0$. In the last row of Table 2 we carry the condition number of collocation matrix.

Table 2

$|y(x) - y_n(x)|$ for the equation (33), $m = 3$

x	$n = 10$		$n = 20$		$n = 40$	
	$r = 1$	$r = 2$	$r = 1$	$r = 2$	$r = 1$	$r = 2$
0.01	0.58E-1	0.17E-2	0.43E-1	0.28E-3	0.24E-1	0.30E-4
0.51	0.99E-3	0.23E-4	0.33E-3	0.94E-5	0.11E-3	0.22E-5
1.	0.42E-3	0.12E-3	0.14E-3	0.12E-4	0.49E-4	0.24E-5
condition number	2.46	2.33	2.44	2.28	2.43	2.26

7. Conclusions

In this paper we have considered the numerical solution of linear Volterra integral equations of the second kind with weakly singular kernel of the form (2) and (3). In order to obtain a sufficiently smooth approximate solution in $(0, X]$, here we have proposed and analyzed a collocation method based on optimal nodal splines.

We remark that the above method could also be applied to obtain the starting values in $[0, T]$, with $T < X$, for another one based on piecewise polynomials on $[T, X]$. Such scheme has been used in [9], with a method based on quasi interpolatory splines defined in [11].

Finally, the generalization of the obtained results to the nonlinear equations would be interesting and its systematic study is under investigation.

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