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SOME INTERPOLATION SCHEMES ON TRIANGLE

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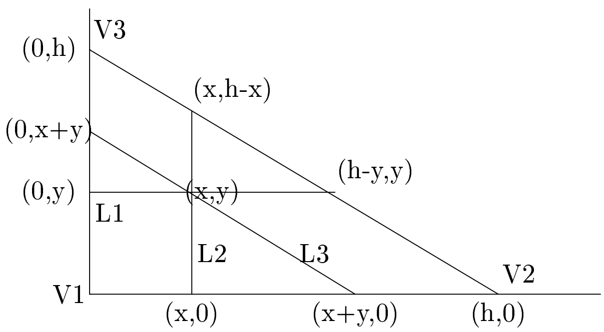
Dedicated to Professor Gheorghe Micula at his 60^{th} anniversary

Abstract. One of the quite simple procedures for constructing multidimensional approximation operators consists in the composition of univatiate approximation operators, using tensor product and boolean sum operations. In this paper, we will construct such interpolation operators for functions defined on a triangle, belonging to $B^r_{pq}(a,b)$ Sard spaces.

1. Introduction

Let $T_h = \left\{(x,y) \in \mathbb{R}^2 | x \geq 0, y \geq 0, x+y \leq h\right\}$ be the standard triangle and $f : T_h \to \mathbb{R}$ a given function. For generating interpolation formulas on this triangle, we will use Lagrange, Hermite or Birkhoff univariate interpolation operators.



In the paper [1], using Lagrange interpolation operators defined by:

$$\begin{array}{rcl} (L_1 f)(x,y) & = & \frac{h-x-y}{h-y} f(0,y) + \frac{x}{h-y} f(h-y,y) \\ (L_2 f)(x,y) & = & \frac{h-x-y}{h-x} f(x,0) + \frac{h}{h-x} f(x,h-x) \\ (L_3 f)(x,y) & = & \frac{x}{x+y} f(x+y,0) + \frac{y}{x+y} f(0,x+y) \end{array} \quad (1)$$

the authors studied some two dimensional discrete and blending interpolation operators. Thus, tensor product of the three operators, $P := L_1 L_2 L_3$, i.e.

$$(Pf)(x,y) = \frac{h-x-y}{h} f(0,0) + \frac{x}{h} f(h,0) + \frac{y}{h} f(0,h)$$

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interpolates the function on the vertices of the triangle. For the remainder term of the interpolation formula:

$$f = Pf + Rf$$

was proved that for $f \in B_{11}(0,0)$

$$(Rf)(x,y) = \frac{x(x-h)}{2} f^{(2,0)}(\xi,0) + \frac{y(y-h)}{2} f^{(0,2)}(0,\eta) + xyf^{(1,1)}(\xi_1,\eta_1)$$

where $\xi, \eta \in [0,h], (\xi_1,\eta_1) \in T_h$.

Also, the boolean sum of every two operators L_i, $i = 1,2,3$ verifies the properties

$$L_i \oplus L_j f|_{\partial T_h} = f|_{\partial T_h}$$

$$L_i \oplus L_j g = g, g \in \mathbb{P}_2^2, \forall i,j = 1,2,3; i \neq j$$

That means that the operator $L_i \oplus L_j$ interpolates the function f on the boundary of T_h and its degree of exactness is 2 ($dex(L_i \oplus L_j) = 2$).

By appropriate select of interpolation operators, we can build interpolation formulas in which the values of the function are interpolated on certain sides and the normal derivatives on others.

For example, if B_1 is the Birkhoff interpolation operator defined by

$$(B_1 f)(x,y) = f(h-y,y) + (x+y-h) f^{(1,0)}(0,y)$$

which interpolates f on the ipotenuza of the triangle T_h and its normal derivative on the cathetus based on Ox, the operator

$$G = B_1 \oplus L_2$$

satisfies the interpolation properties:

$$(Gf)(x,0) = f(x,0), x \in [0,h]$$

$$(Gf)(h-y,y) = f(h-y,y), y \in [0,h]$$

$$(Gf)^{(1,0)}(0,y) = f^{(1,0)}(0,y), y \in [0,h]$$

and $dex(G) = 2$.

For the remainder of the formula $f = Gf + Rf$, it was proved that for $f \in C^{1,2}(T_h)$ and $f^{(0,3)}(0,y)$, $y \in [0,h]$ exist and is continuous, then

$$\|Rf\|_{L_\infty(T_h)} \leq \frac{h^3}{27} \left[\frac{2}{3} \left\| f^{(0,3)}(0,\cdot) \right\|_{L_\infty[0,h]} + \frac{1}{2} \left\| f^{(1,2)} \right\|_{L_\infty(T_h)} \right]$$

2. Next, we will build new interpolation operators for which we will determine the interpolation properties and degree of exactness. Also, the generated interpolation formulas will be studied.

2.1. Let us consider for the beginning the Taylor operator T_1^y defined by

$$(T_1^y f)(x,y) = f(x,0) + yf^{(0,1)}(x,0)$$

which interpolates the function f and its normal derivative $f^{(0,1)}$ with regard to the variable y on the Ox cathetus, respectively the operator L_1^x given in (1), i.e.

$$(L_1^x f)(x,y) = \frac{h-x-y}{h-y} f(0,y) + \frac{x}{h-y} f(h-y,y)$$

Let P_1 be

$$P_1 := L_1^x \oplus T_1^y$$

and

$$f = P_1 f + R_1 f \tag{2}$$

approximation formula generated by P_1.

Theorem 1. *Let consider* $f : T_h \to \mathbb{R}$*. If there exists* $f^{(0,1)}(x,0)$*,* $x \in [0,h]$ *and* $f^{(1,0)}(h,0)$ *then* $P_1 f$ *verifies the interpolation properties:*

$$(P_1 f)(0,y) = f(0,y), y \in [0,h]$$

$$(P_1 f)(h-y,y) = f(h-y,y), y \in [0,h]$$

$$(P_1 f)^{(0,1)}(x,0) = f^{(0,1)}(x,0), x \in [0,h]$$

and $dex(P_1) = 3$*.*

Proof.

$$\begin{aligned} (P_1 f)(x,y) &= \tfrac{h-x-y}{h-y} f(0,y) + \tfrac{x}{h-y} f(h-y,y) + f(x,0) + \\ &+ y f^{(0,1)}(x,0) - \tfrac{h-x-y}{h-y} f(0,0) - \tfrac{x}{h-y} f(h-y,0) - \\ &- \tfrac{y(h-x-y)}{h-y} f^{(0,1)}(0,0) - \tfrac{xy}{h-y} f^{(0,1)}(h-y,0) \end{aligned} \tag{3}$$

Now, the interpolation properties are easy verified, by direct computation.

So, $P_1 f$ coincides with f on a cathetus and the ipotenuza and the normal derivatives concides on the other cathetus.

We, also, have

$$P_1 e_{ij} = e_{ij} \text{ for } i,j \in \mathbb{N}, i+j \le 3 \text{ and } P_1 e_{22} \ne e_{22},$$

where $e_{ij}(x,y) = x^i y^j$. As P_1 is linear, it follows that $dex(P_1) = 3$.

Theorem 2. *If* $f \in B_{2,2}(0,0)$ *then*

$$(R_1 f)(x,y) = \int\int_{T_h} \varphi_{22}(x,y,s,t) f^{(2,2)}(s,t)\, ds dt$$

where $\varphi_{22}(x,y,s,t) = R_1\left[\frac{(x-s)_+^2}{2}\frac{(y-t)_+^2}{2}\right] = \frac{(x-s)_+^2}{2} \cdot \frac{(y-t)_+^2}{2}$

Furthermore, if $f^{(2,2)} \in C(T_h)$ *then*

$$(R_1 f)(x,y) = \frac{1}{36} x y^3 (x+y-h)(h+x-y) f^{(2,2)}(\xi,\eta), (\xi,\eta) \in T_h \tag{4}$$

Proof. As $dex(P_1) = 3$ it results, from the Peano's theorem, that

$$\begin{aligned} (R_1 f)(x,y) &= \int_0^h \varphi_{40}(x,y,s) f^{(4,0)}(s,0)\, ds + \int_0^h \varphi_{04}(x,y,t) f^{(0,4)}(0,t)\, dt + \\ &+ \int_0^h \varphi_{31}(x,y,s) f^{(3,1)}(s,0)\, ds + \int_0^h \varphi_{13}(x,y,t) f^{(1,3)}(0,t)\, dt + \\ &+ \int\int_{T_h} \varphi_{22}(x,y,s,t) f^{(2,2)}(s,t)\, ds dt \end{aligned}$$

Since $\varphi_{40}, \varphi_{31}, \varphi_{04}, \varphi_{13} = 0$ one obtain the first expression of the remainder term.

φ_{22} don't change the sign on T_h. Then, by the Mean theorem the expression (4) follows.

2.2. Now, let T_1 be defined by

$$\left(T_1^x f\right)(x,y) = f(0,y) + x f^{(1,0)}(0,y)$$

which interpolate f and its normal derivatives with regard to the variable x on the Oy cathetus.

Let be

$$P_2 = T_1^x \oplus T_1^y$$

and

$$f = P_2 f + R_2 f$$

the approximation formula generated by the operator P_2.

Theorem 3. *If* $f : T_h \to \mathbb{R}$ *and exist* $f^{(1,0)}_{(x,0)}, f^{(0,1)}_{(0,y)}$, $x, y \in [0,h]$ *then*

1. $P_2 f = f$ *on* ∂T_h.

2. $dex\,(P_2) = 3$.

Proof.

$$\begin{aligned}(P_2 f)(x,y) &= f(x,0) + y f^{(0,1)}(x,0) + f(0,y) + x f^{(1,0)}(0,y) - f(0,0) - \\ &- y f^{(0,1)}(0,0) - x\left[f^{(0,1)}(0,0) + y f^{(1,1)}(0,0)\right]\end{aligned}$$

The first statement results by a direct computation.

Also by direct computation, we obtain $P_2 e_{ij} = e_{ij}$ for $i + j \le 3$ and $P_2 e_{22} \ne e_{22}$, which implies that $dex\,(P_2) = 3$.

Theorem 4. *If* $f \in B_{22}(0,0)$ *then*

$$(R_2 f)(x,y) = \int\int_{T_h} \varphi_{22}(x,y,s,t)\, f^{(2,2)}(s,t)\, dsdt$$

where $\varphi_{22}(x,y,s,t) := R_2\left[\frac{(x-s)_+^2}{2} \frac{(y-t)_+^2}{2}\right] = \frac{(x-s)_+^2}{2} \cdot \frac{(y-t)_+^2}{2}$.

Furthermore, if $f^{(2,2)} \in C(T_h)$ *then*

$$(R_2 f)(x,y) = \frac{x^3 y^3}{36} f^{(2,2)}(\xi,\eta),\ (\xi,\eta) \in T_h.$$

Proof. Knowing that $dex\,(P_2) = 3$ it results, from the Peano's theorem, that

$$\begin{aligned}(R_2 f)(x,y) &= \int_0^h \varphi_{40}(x,y,s)\, f^{(4,0)}(s,0)\, ds + \int_0^h \varphi_{31}(x,y,s)\, f^{(3,1)}(s,0)\, ds + \\ &+ \int_0^h \varphi_{04}(x,y,t)\, f^{(0,4)}(0,t)\, dt + \int_0^h \varphi_{13}(x,y,t)\, f^{(1,3)}(0,t)\, dt + \\ &+ \int\int_{T_h} \varphi_{22}(x,y,s,t)\, f^{(2,2)}(s,t)\, dsdt\end{aligned}$$

Since $\varphi_{40}, \varphi_{31}, \varphi_{04}, \varphi_{13} = 0$ it results the first expression of the remainder term.

φ_{22} don't change the sign on T_h.Then, by the Mean theorem follows the second expression of the remainder term.

2.3. At last, let us consider the univariate operators B_1^x and B_1^y defined respectively by

$$(B_1^x)(x,y) = f(0,y) + xf^{(1,0)}(h-y,y)$$

and

$$(B_1^y)(x,y) = f(x,0) + yf^{(0,1)}(x,h-x)$$

The goal is to study the operator $P_3 := B_1^x \oplus B_1^y$ i.e.

$$\begin{aligned}(P_3f)(x,y) &= f(x,0) + f(0,y) + xf^{(1,0)}(h-y,y) + yf^{(0,1)}(x,h-x) - \\ &- f(0,0) - xf^{(1,0)}(h-y,0) - yf^{(0,1)}(0,h) - \\ &- xy\left(f^{(1,1)} - f^{(0,2)}\right)(h-y,y)\end{aligned}$$

Theorem 5. *If $f : T_h \to \mathbb{R}$ and there exists the derivatives $f^{(1,0)}(h-y,y)$, $f^{(0,1)}(x,h-x)$, $f^{(1,1)}(h-y,y)$, $f^{(0,2)}(h-y,y)$ and $f^{(1,0)}(h-y,0)$ for $x,y \in [0,h]$ than P_3 exists and*

$$(P_3f)(x,0) = f(x,0)$$

$$(P_3f)(0,y) = f(0,y)$$

$$(P_3f)^{(1,0)}(h-y,y) = f^{(1,0)}(h-y,y), x,y \in [0,h]$$

and

$$dex(P_3) = 2.$$

Proof. The first statement follows by a straightforward computation. Also, it is easy to verify that $P_3e_{ij} = e_{ij}$ for all $i,j \in \mathbb{N}$ with $i+j \le 2$ and, for example $P_3e_{21} \ne e_{21}$. So, $dex(P_3) = 2$.

For the remainder term of the interpolation formula

$$f = P_3f + R_3f$$

we have:

Theorem 6. *If $f \in B_{12}(0,0)$ then*

$$\begin{aligned}(R_3f)(x,y) &= \frac{1}{6}y\left[y^2 + 6x(h-x-y)\right]f^{(0,3)}(0,\eta) - \\ &- \frac{1}{2}xy(2h+2x-y)f^{(1,2)}(\xi_1,\eta_1)\end{aligned}$$

Proof. As $dex(P_3) = 2$, using the Peano's theorem, one obtain

$$\begin{aligned}(R_3f)(x,y) &= \textstyle\int_0^h \varphi_{30}(x,y,s)f^{(3,0)}(s,0)\,ds + \int_0^h \varphi_{21}(x,y,s)\cdot \\ &\cdot\ f^{(2,1)}(s,0)\,ds + \textstyle\int_0^h \varphi_{03}(x,y,t)f^{(0,3)}(0,t)\,dt + \\ &+ \textstyle\int\int_{T_h} \varphi_{12}(x,y,s,t)f^{(1,2)}(s,t)\,dsdt\end{aligned} \tag{5}$$

But, $\varphi_{30} = 0$, $\varphi_{21} = 0$, $\varphi_{03} \geq 0$ and $\varphi_{12} \leq 0$ on T_h. Using the mean theorem we have

$$\begin{aligned}(R_3 f)(x,y) &= f^{(0,3)}(0,\eta)\int_0^h \varphi_{03}(x,y,t)\,dt + \\ &+ f^{(1,2)}(\xi_1,\eta_1)\int\int_{T_h}\varphi_{12}(x,y,s,t)\,dsdt\end{aligned}$$

and (5) follows.

References

[1] Barnhill, R.E., Birkhoff, G., Gordon, W.J., *Smooth interpolation in triangle*, J. Approx. Theory 8, 1973, 114-128.

[2] Barnhill, R.E., Mansfield, *Sard Kernel theorems on triangular and rectangular domains with extensions and applications to finite element error*, Technical Report 11, Dep. of Math., Brunel Univ., 1972.

[3] Blaga, P., Coman, Gh., *Multivariate interpolation formulas of Birkhoff type*, Studia Univ. Babes-Bolyai, Mathematica 26, 1981, 14-22.

[4] Bohner, K., Coman, Gh., *Blending interpolation schemes on triangle with error bounds*, Springer Verlag, Berlin - Heidelberg - New York, 1977, 14-37.

[5] Coman, Gh., Tâmbulea, L., Gânscă, I., *Multivariate approximation*, Seminar on Numerical and Statistical Calcules, Preprint nr. 1, 1996, 29-60.

[6] Sard, A., *Linear Approximation*, American Mathematical Society, Providence, Rhode Island, 1963.

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