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MINIMUM VALUE OF A MATRIX NORM WITH APPLICATIONS TO MAXIMUM PRINCIPLES FOR SECOND ORDER PARABOLIC SYSTEMS

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Dedicated to Professor Gheorghe Micula at his 60^{th} *anniversary*

Abstract. The purpose of this paper is to use an estimation of minimum value of a matrix norm to improve some maximum principles given by I.A. Rus in 1968.

1. **Introduction**

Let M be the linear space of vectorial functions $u = (u_1(x,t), ..., u_n(x,t))$ which belongs to $C(\Omega)$ and are twice continuous differentiable in x and continuous differentiable in t. $\Omega \subseteq \mathbb{R}^2$ is a bounded domain. In M we consider the following system:

$$Lu := p^2 I_n \frac{\partial^2 u}{\partial x^2} + B\frac{\partial u}{\partial x} + Cu - I_n\frac{\partial u}{\partial t} = 0 \tag{1}$$

where $p \in \mathbb{R}^*, B = (b_{ij}(x,t)), C = (c_{ij}(x,t))$ are squared matrixes defined on Ω .

Let $P_o(x_o, t_o) \in \Omega$. We will denote by $S(P_o)$ the set of points Q for which there exist an arch on which the ordinate t is non-decreasing beginning with the point Q.

There are some maximum principles for the solution of system (1) (see for example [2] and [3]).

Let $u = u(x,t)$ be a solution of the system (1). In [3] the following principle is given:

Theorem 1. *Suppose that for each* $(x,t) \in \Omega$, *there exist* $\widetilde{\beta}(x,t) \in \mathbb{R}$ *such that:*

$$\xi \begin{pmatrix} -p^2 I_n & 0 \\ B(x,t) - \widetilde{\beta}(x,t) I_n & C(x,t) \end{pmatrix} \xi^* < 0, \forall \xi \in \mathbb{R}^{2n}, \xi \neq 0 \tag{2}$$

If $R(x,t) := \left(\sum_{i=1}^{n} u_i^2\right)^{1/2}$ *attains his maximum in* $P_o \in \Omega$, *then* $R(Q) = R(P_o)$, *for each* $Q \in S(P_o)$.

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Remark 1. *If, for each* $(x,t)\in\Omega$, *there exist* $\widetilde{\beta}(x,t)\in\mathbb{R}$ *and* $\varepsilon(x,t)>0$ *such that:*

(i) $\xi C(x,t)\xi^* < -\left(\frac{\varepsilon(x,t)}{p}\right)^2\|\xi\|^2, \forall\xi\in\mathbb{R}^n, \xi\neq 0$;

(ii) $\left\|B(x,t)-\widetilde{\beta}(x,t)I_n\right\|_2 \leq 2\varepsilon(x,t)$,

where $\|\cdot\|_2$ *is the spectral norm, then (2) holds.*

The aim of this paper is to give some conditions which imply (ii).
Let $A\in M_n(\mathbb{R})$, J the Jordan normal form of A. We know that there exist a nonsingular matrix T such that $A=TJT^{-1}$.

We shall denote:

$$\begin{aligned}\widetilde{\alpha} &= \begin{cases}\frac{1}{n}\sum\limits_{k=1}^{s} n_k\lambda_k, \lambda_k\in\mathbb{R}\\ \frac{1}{n}\sum\limits_{k=1}^{s} n_k\ Re\lambda_k, \lambda_k\in\mathbb{C}\backslash\mathbb{R}\end{cases}\\ \gamma_F &= \|T\|_F\cdot\left\|T^{-1}\right\|_F\\ m_F &= \|J-\widetilde{\alpha}I_n\|_F\end{aligned}$$

where λ_k are the eigenvalues of A, n_k is the number of λ_k which appears in Jordan blocks (generated by λ_k) and $\|\cdot\|_F$ is the euclidean norm of a matrix (see [1]).

We shall use the following result given in [1]:

Theorem 2. *Let* $\varphi_{\|\cdot\|}:\mathbb{R}\to\mathbb{R}, \varphi_{\|\cdot\|}(\alpha)=\|A-\alpha I_n\|$, $\|\cdot\|$ *being one of the following norms:* $\|\cdot\|_F$, $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$. *In these conditions:*

$$\varphi_{\|\cdot\|}(\widetilde{\alpha})\leq\sqrt{n}\gamma_F m_F$$

In section 2 of this paper we shall give the main result in case of system 1 and in section 3, using the same instrument, we shall try to improve a maximum principle in case of elliptic-parabolic systems.

2. Main result in parabolic case

Using Theorem 2 and choosing $\varepsilon(x,t)=\frac{1}{2}\sqrt{n}\gamma_F m_F$, Theorem 1 becomes:

Theorem 3. *Suppose that* $\xi C(x,t)\xi^* < -\frac{1}{4p^2}n\gamma_F^2 m_F^2\|\xi\|^2, \forall\xi\in\mathbb{R}^n, \xi\neq 0$, $\forall(x,t)\in\Omega$. *If* $R(x,t)=\left(\sum\limits_{i=1}^{n}u_i^2\right)^{1/2}$ *attains his maximum in* $P_o\in\Omega$, *then* $R(Q)=R(P_o)$, *for each* $Q\in S(P_o)$.

Example 1. *Let us consider the system (1) with* $B=\begin{pmatrix}a_1 & a_2\\ a_3 & a_1\end{pmatrix}$ *and without restraining the generality we shall suppose that* $a_2, a_3>0$. *In this case we shall have:* $\widetilde{\beta}=a_1$, $\varepsilon=a_2+a_3$ *and:*

$$\|B-a_1I_2\|_2\leq\|B-a_1I_2\|_F=\sqrt{a_2^2+a_3^2}<2(a_2+a_3)=\sqrt{2}\gamma_F m_F=2\varepsilon$$

$$\xi C(x,t)\xi^* < -\frac{1}{p^2}(a_2+a_3)^2\|\xi\|^2 \tag{3}$$

So if (3) holds than we have:

$$\xi \begin{pmatrix} -p^2 I_2 & 0 \\ B - \widetilde{\beta} I_2 & C \end{pmatrix} \xi^* < \frac{1}{4p^2} \left[a_2^2 + a_3^2 - 4\left(a_2 + a_3\right)^2 \right] \left\| \xi' \right\|^2 < 0$$

where $\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4, \xi \neq 0, \xi' = (\xi_3, \xi_4) \in \mathbb{R}^2, \xi' \neq 0.$

3. **Elliptic-parabolic case**

Let us consider now the following system:

$$Lu := \frac{\partial^2 u}{\partial x^2} + y^p \frac{\partial^2 u}{\partial y^2} + A \frac{\partial u}{\partial x} + B \frac{\partial u}{\partial y} + Cu = 0 \tag{4}$$

where L is defined in $M = C^{2,n}(\Omega) \cap C^{0,n}\left(\overline{\Omega}\right)$, $A = (a_{ij}(x,y)), B = (b_{ij}(x,y))$, $C = (c_{ij}(x,y))$ are squared matrixes defined on $\overline{\Omega}$, $p \in \mathbb{R}_+$.

Ω is a domain included in the half-plan $y > 0$ and which has a part of frontier laying on $y = 0$, between the points $P(0,0)$ and $Q(1,0)$. The operator L is elliptic in Ω and parabolic on $\widehat{PQ}$.

Let $u \in C^2(\Omega, \mathbb{R}^n) \cap C\left(\overline{\Omega}, \mathbb{R}^n\right), u = u(x,y)$, be a solution of (3) and

$$R(x,y) := \left(\sum_{i=1}^{n} u_i^2 \right)^{1/2}.$$

Theorem 4. *([3])If:*

1. *for each $(x,y) \in \Omega$, there exist $\widetilde{\alpha}(x,y), \widetilde{\beta}(x,y) \in \mathbb{R}$ such that*
$$\xi \begin{pmatrix} -I_n & 0 & 0 \\ 0 & -y^p I_n & 0 \\ A(x,y) - \widetilde{\alpha}(x,y) I_n & B(x,y) - \widetilde{\beta}(x,y) I_n & C(x,y) \end{pmatrix} \xi^* < 0, \tag{5}$$
for all $\xi \in \mathbb{R}^{3n}, \xi \neq 0$;
2. *B is symmetric such that if $\lambda_1(x,y)$ is the first eigenvalue, then $\lambda_1(x,0) > 0$;*
3. *u is a regular solution of (3) and $R > 0$ in Ω;*
4. *$\lim\limits_{y \to 0} \frac{\partial R(x,y)}{\partial y}$ exist and is bounded,*

then $R = R(x,y)$ cannot attain his maximum value on $\widehat{PQ}$(open).

Remark 2. *If, for each $(x,y) \in \Omega$, there exist $\widetilde{\alpha}(x,y), \widetilde{\beta}(x,y) \in \mathbb{R}$ and $\varepsilon_1(x,y), \varepsilon_2(x,y) > 0$ such that:*

$$(i) \xi C(x,y) \xi^* < -\left(\varepsilon_1^2(x,y) + y^{-p} \varepsilon_2^2(x,y)\right) \|\xi\|^2, \forall \xi \in \mathbb{R}^n, \xi \neq 0;$$

$$(ii) \left\| A(x,y) - \widetilde{\alpha}(x,y) I_n \right\|_2 \leq 2\varepsilon_1(x,y), \ \left\| B(x,y) - \widetilde{\beta}(x,y) I_n \right\|_2 \leq 2\varepsilon_2(x,y),$$

then (5) holds.

Using Theorem 2 and choosing $\varepsilon_1 = \frac{1}{2}\sqrt{n}\gamma_F^A m_F^A$ and $\varepsilon_2 = \frac{1}{2}\sqrt{n}\gamma_F^B m_F^B$, the remark from above becomes:

Remark 3. *If:*

$$\xi C(x,y)\xi^* < -\frac{1}{4}n\left[\left(\gamma_F^A m_F^A\right)^2 + \left(\gamma_F^B m_F^B y^{-\frac{p}{2}}\right)^2\right]\|\xi\|^2, \forall \xi \in \mathbb{R}^n, \xi \neq 0, \forall (x,y) \in \Omega$$

then (5) holds.

Example 2. *Let us consider the system (2) with* $A = B = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_1 \end{pmatrix}$.

For $\widetilde{\alpha} = \widetilde{\beta} = a_1$ and $a_2, a_3 > 0$, we have $\varepsilon_1 = \varepsilon_2 = a_2 + a_3$, $A - a_1 I_2 = B - a_1 I_2 = \begin{pmatrix} 0 & a_2 \\ a_3 & 0 \end{pmatrix}$.

If $\xi C(x,y)\xi^* < -(a_2+a_3)^2(1+y^{-p})\|\xi\|^2$, then:

$$\xi \begin{pmatrix} -I_2 & 0 & 0 \\ 0 & -y^p I_2 & 0 \\ A - \widetilde{\alpha} I_2 & B - \widetilde{\beta} I_2 & C \end{pmatrix} \xi^* < \frac{1}{4}[a_2^2 + a_3^2 - 4(a_2+a_3)^2](1+y^{-p})\|\xi'\|^2 < 0$$

where $\xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) \in \mathbb{R}^6, \xi \neq 0, \xi' = (\xi_5, \xi_6) \in \mathbb{R}^2, \xi' \neq 0$.

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