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ABOUT SOME VOLTERRA PROBLEMS SOLVED BY A PARTICULAR SPLINE COLLOCATION

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Dedicated to Professor Gheorghe Micula at his 60^{th} *anniversary*

Abstract. In this paper we propose a deficient spline collocation method for a special Volterra integral equation problem. The existence and uniqueness of the approximating spline are investigated. Some numerical examples illustrate the efficiency of the proposed numerical method.

1. **Introduction**

The theory and applications of the Volterra integral equations of the form

$$y(x) = \int_0^x K(x,t,y(t))dt + g(x), \quad x \in [0,T]$$

is an important subject within applied mathematics. Volterra integral equations are used as mathematical models for many and varied physical phenomena and processes but they occur as reformulations of other mathematical problems.

In the present work we consider the following Volterra equation with constant delay $\tau > 0$:

$$y(x) \ = \int_0^x K_1(x,t,y(t))dt + \int_0^{x-\tau} K_2(x,t,y(t))dt + g(x), x \in J = [0,T] \qquad (1)$$

with $y(x) = \varphi(x)$, $x \in [\tau, 0)$.

Equation (1) is worth studying as it is frequently encountered in physical and biological modeling processes (e.g. [5]).

We assume that the given functions

$\varphi : [-\tau, 0] \to \mathbb{R}, g : J \to \mathbb{R}, K_1 : \Omega \times \mathbb{R} \to \mathbb{R}(\Omega := [(x,t) : 0 \leq t \leq x \leq T]),$
$K_2 : \Omega_\tau \times \mathbb{R} \to \mathbb{R}(\Omega_\tau := J \times [-\tau, T-\tau])$

are at least continuous on their domains such that (1) possesses a unique solution $y \in C(J)$, and $\varphi \in C^{m-2}[-\tau, 0]$, $g \in C^{m-2}[0,T]$.

In the following, let us assume in (1) that :

1. K_1 satisfies the following Lipschitz condition :

$$\| K_1(x,t,y_1) - K_1(x,t,y_2) \| \leq L_1 \| y_1 - y_2 \|$$

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$$\forall (x,t,y_1),(x,t,y_2) \in J \times J \times \mathbb{R}, t \le x$$

with $L_1 \ge 0$ a constant independent of x and t.

2. K_2 satisfies the following Lipschitz condition :

$$\| K_2(x,t,z_1) - K_2(x,t,z_2) \| \le L_2 \| z_1 - z_2 \|$$

$$\forall (x,t,z_1),(x,t,z_2) \in J \times J \times \mathbb{R}, t \le x$$

with $L_2 \ge 0$ also a constant independent of x and t.

Recently, various aspects of numerical methods for (1) have been studied from the point of view of polynomial collocation methods (e.g. [1], [2], [8]). In this context here we propose to approximate the solution of (1) by means of functions pertaining to the class of splines $s: J \to \mathbb{R}$,($s \in \mathcal{S}_m$, $s \in C^{m-2}$) of degree $m \ge 2$ and deficiency $d \ge 2$.

We already used an analogous deficient spline collocation method in the case of delay differential equations [3], [4]. As it revealed simple and efficient, here we propose to extend it to Volterra integral equations with delay argument to provide an alternative to the discrete collocation method proposed in [1]. Indeed our method presents some advantages:

- it does not require any additional initial value
- it provides a global approximation of the solution
- in case of need, the lenght of each collocation step can be modified, and similarly the degree m of the used spline functions and deficiency d
- the proposed numerical method reveals extremely easy to be implemented in the linear case.

We emphasize that this method is peculiar for solutions belonging to low regularity class. Indeed we use $m = 2$ or $m = 3$ only; so that the used splines are $s \in C^0$ or at most $s \in C^1$; there is numerical evidence that it suffices in order to approximate solutions belonging to class C^0 or C^1.

In Section 2 we present the numerical method to approximate the solution by collocation of deficient spline functions; Section 3 is devoted to theoretical results referring to existence and unicity of the numerical solution and there we recall also some results about convergence and numerical stability. In the last Section we report some examples relating to integral equations with solutions characterized by low regularity.

2. **Construction of approximating spline solution**

In this section we describe the numerical model used to approximate the solution of (1).

Firstly we shall construct a polynomial spline function of degree $m > 1$, which we denote by s. On the interval $J := [0,T]$ the spline function s is defined in $[t_k, t_{k+1}]$ where $t_k := t_0 + kh$, $k = 0,1,\cdots,N$; $t_0 := 0$, $t_N = T$, $h := \frac{T}{N}$ as :

$$s_k(t) := \sum_{j=0}^{m-2} \frac{s_{k-1}^{(j)}(t_k)}{j!}(t-t_k)^j + \frac{a_k}{(m-1)!}(t-t_k)^{m-1} + \frac{b_k}{m!}(t-t_k)^m$$

We choose to determine coefficients a_k, b_k by the following system of collocation conditions:

$$
\begin{aligned}
&s_k(t_k+\frac{h}{2})=\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}K_1(t_k+\frac{h}{2},t,s_j(t))dt+\int_{t_k}^{t_k+\frac{h}{2}}K_1(t_k+\frac{h}{2},t,s_k(t))dt+\\
&+\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}-\tau}K_2(t_k+\frac{h}{2},t,s_{j-1}(t))dt+\int_{t_k}^{t_k+\frac{h}{2}-\tau}K_2(t_k+\frac{h}{2},t,s_{k-1}(t))dt+g(t_k+\frac{h}{2})\\
&s_k(t_{k+1})=\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}K_1(t_{k+1},t,s_j(t))dt+\int_{t_k}^{t_{k+1}}K_1(t_{k+1},t,s_k(t))dt+\\
&+\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}-\tau}K_2(t_{k+1},t,s_{j-1}(t))dt+\int_{t_k}^{t_{k+1}-\tau}K_2(t_{k+1},t,s_{k-1}(t))dt+g(t_{k+1})
\end{aligned}
\tag{2}
$$

provided that

$$s_{-1}(0)=y(0)=\varphi(0),s'_{-1}(0)=y'(0)=\varphi'(0),\cdots,s_{-1}^{(m-2)}(0)=y^{(m-2)}(0)=\varphi^{(m-2)}(0)$$

Our model is thus reduced to compute the solution of the non-linear system (2), through which the spline is globally determined on the interval J.

3. **The theoretical results**

It remains to prove that for h sufficiently small, the parameters a_k, b_k, $0\le k\le N-1$ can be uniquely determined from (2).

Theorem. *Let as consider the Volterra equation* (1). *If* K_1 *and* K_2 *satisfy the hypotheses* 1 *and* 2, *and if* h *is small enough, then there exists a unique spline solution* s *of* (1) *given by the above construction.*

Proof. If we set

$$A_k(t)=\sum_{j=0}^{m-2}\frac{s_{k-1}^{(j)}(t_k)}{j!}(t-t_k)^j$$

then (2) becomes:

$$a_k = \left(\frac{2}{h}\right)^{m-1}(m-1)![-A_k(t_k+\frac{h}{2}) - \frac{b_k}{m!}\left(\frac{h}{2}\right)^m +$$
$$+\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}} K_1(t_k+\frac{h}{2}, t, A_j(t) + \frac{a_j}{(m-1)!}(t-t_j)^{m-1} + \frac{b_j}{m!}(t-t_j)^m)dt +$$
$$+\int_{t_k}^{t_k+\frac{h}{2}} K_1(t_k+\frac{h}{2}, t, A_k(t) + \frac{a_k}{(m-1)!}(t-t_k)^{m-1} + \frac{b_k}{m!}(t-t_k)^m)dt +$$
$$+\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}-\tau} K_2(t_k+\frac{h}{2}, t, A_{j-1}(t) + \frac{a_{j-1}}{(m-1)!}(t-t_{j-1})^{m-1} + \frac{b_{j-1}}{m!}(t-t_{j-1})^m)dt +$$
$$+\int_{t_k}^{t_k+\frac{h}{2}-\tau} K_2(t_k+\frac{h}{2}, t, A_{k-1}(t) + \frac{a_{k-1}}{(m-1)!}(t-t_{k-1})^{m-1} + \frac{b_{k-1}}{m!}(t-t_{k-1})^m)dt + g(t_k+\frac{h}{2})]$$
$$b_k = \frac{m!}{h^m}[-A_k(t_{k+1}) - a_k\frac{h^{m-1}}{(m-1)!} +$$
$$+\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}} K_1(t_{k+1}, t, A_j(t) + \frac{a_j}{(m-1)!}(t-t_j)^{m-1} + \frac{b_j}{m!}(t-t_j)^m)dt +$$
$$+\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}-\tau} K_2(t_{k+1}, t, A_{j-1}(t) + \frac{a_{j-1}}{(m-1)!}(t-t_{j-1})^{m-1} + \frac{b_{j-1}}{m!}(t-t_{j-1})^m)dt + g(t_{k+1})]$$

thus we can deduce

$$a_k = F_1(a_k, b_k)$$
$$b_k = F_2(a_k, b_k)$$

where F_1 and F_2 are the right hand side of the above equations.

Now we define the application $F : \mathbb{R}^2 \to \mathbb{R}^2$ as $(a_k, b_k) \to F(a_k, b_k) := (F_1(a_k, b_k), F_2(a_k, b_k))$ and

$$d(F(a'_k, b'_k), F(a''_k, b''_k)) := |F_1(a'_k, b'_k) - F_1(a''_k, b''_k)| + |F_2(a'_k, b'_k) - F_2(a''_k, b''_k)|$$

At first, for $m > 1$, we have

$$|F_1(a'_k, b'_k) - F_1(a''_k, b''_k)| \le \frac{hL_1}{m}(2|a'_k - a''_k| + \frac{3h}{2(m+1)}|b'_k - b''_k|)$$

and similarly

$$|F_2(a'_k, b'_k) - F_2(a''_k, b''_k)| \le L_1(3|a'_k - a''_k| + \frac{5h}{2(m+1)}|b'_k - b''_k|)$$

and taking account that

$$|F_2(a'_k, b'_k) - F_2(a''_k, b''_k)| = |F_2(F_1(a'_k, b'_k), b'_k) - F_2(F_1(a''_k, b''_k), b''_k)|$$

from the previous relations at last it follows that

$$\begin{aligned} d(F(a'_k, b'_k), F(a''_k, b''_k)) &\leq hL_1(\tfrac{2+6L_1}{m}|a'_k - a''_k| + \\ + \tfrac{3h+9L_1h+5m}{2m(m+1)}|b'_k - b''_k|) &\leq MhL_1 d((a'_k, b'_k), (a''_k, b''_k)) \end{aligned}$$

where $M = \max\{(1+3L_1), \frac{1}{6}(\frac{3h}{2} + \frac{9h}{2}L_1 + 5)\}$. The upper bound was obtained using $m = 2$.

Therefore, for $MhL_1 < 1$, that is $h < \frac{1}{ML_1}$, F is a contraction and system (2) has a unique solution, which can be found by iterative method.

It is worth noting that these conditions can be greatly simplified for the linear case when in (1) we have $K_1(x, t, y(t)) = k_1(x, t)y(t)$ and $K_2(x, t, y(t)) = k_2(x, t)y(t)$; this case can be treated in a very simple and efficient way.

About the convergence and the numerical stability, we recall results presented in [6], where the case of integral equations without delay arguments is studied. The comprehensive investigation of the convergence will be approached elsewhere.

4. **Numerical examples**

In the following we present some numerical results to enlighten the features of the proposed numerical method. We emphasize that we will show examples just for the linear case and with exact solution belonging to a low regularity class, because our method is dedicated just to these cases, even though it works also for general cases.

Our computer programs are written in MATLAB5.3, which has a machine precision $\varepsilon \simeq 10^{-16}$.

Example 1.

Consider the following integral equation with delay arguments:

$$\begin{aligned} y(x) &= g(x) + \int_0^x y(s)ds - \int_0^{x-\tau} y(s)ds \\ \tau &= 1, \quad y(x) = 0 \text{ for } x \in [-1, 0] \\ g(x) &= \begin{cases} x - \frac{x^2}{2} & \text{for} \quad x \in [0, 1/2] \\ \frac{x^2}{2} - 2x + \frac{5}{4} & \text{for} \quad x \in [1/2, 1] \end{cases} \end{aligned}$$

The exact solution is:

$$y(x) = \begin{cases} x & \text{for} \quad x \in [0, 1/2] \\ 1 - x & \text{for} \quad x \in [1/2, 1] \end{cases}$$

where $y \in C^0[0, 1]$.

Using $m = 2$ and $d = 2$, we built spline $s \in C^0$. With integration step $h = 0.5$, we obtain numerical results with an error of order 10^{-15}, which means that in practice our results are exact within the machine precision.

Figure 1 refers just to the case $h = 0.5$; there solid line shows the exact solution in $[0, 1]$ together with the history in $[-1, 0]$; squares show the integration points and circles show intermediate points of the numerical solution computed by means of the analytical expression of spline relating to each integration interval.

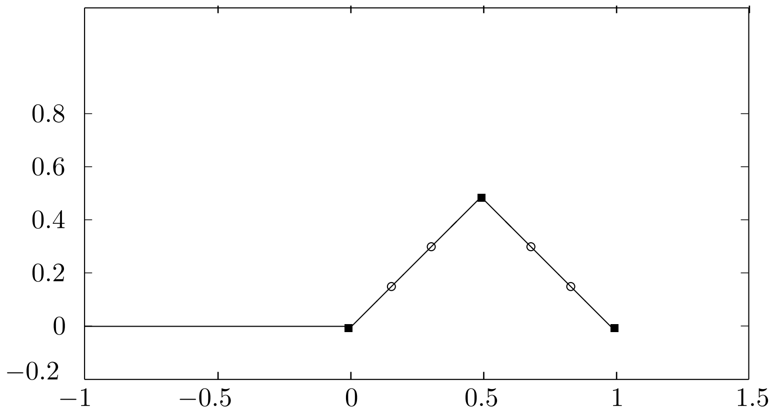

Fig. 1

It is evident that the numerical solution coincides with the exact solution.

Example 2.

Consider the following integral equation with delay arguments:

$$\begin{aligned} y(x) &= g(x) + \int_0^x y(s)ds - \int_0^{x-\tau} y(s)ds \\ \tau &= 1, \quad y(x) = 0 \text{ for } x \in [-1, 0] \\ g(x) &= \begin{cases} 100x - 50x^2 & \text{for} \quad x \in [0, 1/2] \\ -400(x-1)^3 + 100(x-1)^4 - \frac{75}{4} & \text{for} \quad x \in [1/2, 1] \end{cases} \end{aligned}$$

The exact solution is:

$$y(x) = \begin{cases} 100x & \text{for} \quad x \in [0, 1/2] \\ -400(x-1)^3 & \text{for} \quad x \in [1/2, 1] \end{cases}$$

Using $m = 2$ and $d = 2$, we built splines $s \in C^0$. Even in this case the solution y to be approximated belongs to class C^0, but it is the linear in the first integration subinterval only. Therefore we used a large integration step $h_1 = 0.5$ in $[0, 1/2]$ and a shorter step h_2 in $[1/2, 1]$.

Figure 2 refers just to the case $h_1 = 0.5$ and $h_2 = 0.125$; there solid line shows the exact solution in $[0, 1]$ together with the history in $[-1, 0]$; rectangles show the integration points and circles show intermadiate points of the numerical solution computed by means of the analytical expression of spline relating just to the first three integration intervals (for graphical convenience).

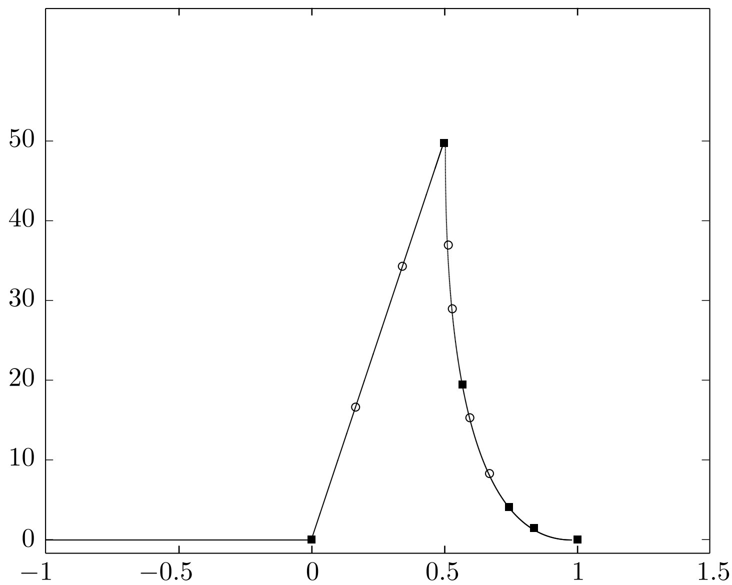

Fig. 2

It is evident that even in this case results are very satisfactory.

In more details, the numerical solution in $x = 1$ is computed with an error equal to $1.0E-2$ when $h_2 = 0.25$ and with an error equal to $6.6E-4$ when $h_2 = 0.125$.

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