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SCHURER-STANCU TYPE OPERATORS

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Dedicated to Professor Gheorghe Micula at his 60^{th} anniversary

Abstract. Considering two non-negative parameters α,β which satisfy $0\leq\alpha\leq\beta$ and a given non-negative integer p, the Stancu-Schurer type operators $\widetilde{S}_{m,p}^{(\alpha,\beta)}:C(0,1+p])\to C([0,1])$

$$\left(\widetilde{S}_{m,p}^{(\alpha,\beta)}f\right)(x)=\sum\nolimits_{k=0}^{m+p}\widetilde{p}_{m,k}(x)f\left(\frac{k+\alpha}{m+\beta}\right)$$

are introduced and some approximation properties of these operators are studied.

1. **Preliminaries**

Let $p\geq 0$ be a given integer. In 1962, F. Schurer (see ([7])), introduced and studied the linear positive operator $\widetilde{B}_{m,p}:C([0,1+p])\to C([0,1])$, defined for any $f\in C([0,1+p])$ and any $m\in\mathbb{N}$ by

$$\left(\widetilde{B}_{m,p}f\right)(x)=\sum\nolimits_{k=0}^{m+p}\widetilde{p}_{mk}(x)f(k/m) \tag{1.1}$$

where $\widetilde{p}_{mk}(x)=\binom{m+p}{k}x^k(1-x)^{m+p-k}$ are the fundamental Schurer polynomials.

Considering the given real parameters α,β which satisfy $0\leq\alpha\leq\beta$, in 1968, D.D. Stancu (see ([9])), constructed the linear positive operators $P_m^{(\alpha,\beta)}:C([0,1])\to C([0,1])$ defined for any $f\in C([0,1])$ and any $m\in\mathbb{N}$ by

$$\left(P_m^{(\alpha,\beta)}f\right)(x)=\sum\nolimits_{k=0}p_{mk}(x)f\left(\frac{k+\alpha}{m+\beta}\right) \tag{1.2}$$

where $p_{mk}(x)=\binom{m}{k}x^k(1-x)^{m-k}$ are the fundamental Bernstein polynomials.

Note that for $p=0$, the operator (1.1) reduces to the classical Bernstein operator and for $\alpha=\beta=0$, the operator (1.2) reduces also to the classical Bernstein operator. Follows that the above operators generalize the classical Bernstein operator.

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Let $\widetilde{S}_{m,p}^{(\alpha,\beta)} : C([0,1+p] \to C([0,1])$ be defined for any $f \in C([0,1+p])$ and any $m \in \mathbb{N}$, by:

$$\left(\widetilde{S}_{m,p}^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right) \tag{1.3}$$

For $\alpha = \beta = 0$ the operator (1.3) reduces to the Schurer operator (1.1) and for $p = 0$, (1.3) reduces to the Stancu operator (1.2).

In what follows the operator defined by (1.3) will be called Schurer-Stancu type operator.

The focus of the paper is to investigate approximation properties of operator (1.3).

2. **Main results**

Lemma 2.1. *The Shurer-Stancu operators, defined by* (1.3)*, are linear and positive.*

Proof. The assertions follows from definition (1.3). □

Like usually, let us to denote by $e_k(s) = s^k, k \in \mathbb{N}$ the test functions.

Lemma 2.2. *For any $x \in [0,1+p]$ and any $m \in \mathbb{N}$ the Schurer-Stancu operators* (1.3) *verify*

$$\left(\widetilde{S}_{m,p}^{(\alpha,\beta)} e_0\right)(x) := \widetilde{S}_{m,p}^{(\alpha,\beta)}(1;x) = 1 \tag{2.1}$$

$$\left(\widetilde{S}_{m,p}^{(\alpha,\beta)} e_1\right)(x) := \widetilde{S}_{m,p}^{(\alpha,\beta)}(s;x) = \frac{m+p}{m+\beta}x + \frac{\alpha}{m+\beta} \tag{2.2}$$

$$\begin{aligned}\left(\widetilde{S}_{m,p}^{(\alpha,\beta)} e_2\right)(x) &= \widetilde{S}_{m,p}^{(\alpha,\beta)}(s^2;x) = \\ &= \frac{1}{(m+\beta)^2}\left\{(m+p)^2x^2 + (m+p)x(1-x) + \right. \\ &+ \left. 2\frac{\alpha m(m+p)}{m+\beta}x + \frac{\alpha^2(3m+\beta)}{m+\beta}\right\}\end{aligned} \tag{2.3}$$

Proof. Using the definition (1.3), we get

$$\widetilde{S}_{m,p}^{(\alpha,\beta)}(1;x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) = \widetilde{B}_{m,k}(x) = \widetilde{B}_{m,p}(1;x) = 1,$$

where we used a well known property of $\widetilde{B}_{m,p}$ (see([7])).

Next

$$\begin{aligned}\widetilde{S}_{m,p}^{(\alpha,\beta)}(s;x) &= \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x)\frac{k+\alpha}{m+\beta} = \\ &= \frac{m}{m+\beta}\sum_{k=0}^{m+\beta} \widetilde{p}_{m,k}(x) \cdot \frac{k}{m} + \frac{\alpha}{m+\beta}\sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) = \\ &= \frac{m}{m+\beta}\widetilde{B}_{m,p}(s;x) + \frac{\alpha}{m+\beta}\widetilde{B}_{m,p}(1;x)\end{aligned}$$

But (see ([7])):

$$\widetilde{B}_{m,p}(s;x) = \left(1 + \frac{p}{m}\right)x, \widetilde{B}_{m,p}(1;x) = 1$$

We can then conclude that

$$\widetilde{S}_{m,p}^{(\alpha,\beta)}(s;x)=\frac{m+\beta}{m+\beta}x+\frac{\alpha}{m+\beta},$$

i.e. (2.2) holds.
In a same way, we obtain

$$\widetilde{S}_{m,p}^{(\alpha,\beta)}(s^2;x)=\sum_{k=0}^{m+p}\widetilde{p}_{m,k}(x)\cdot\left(\frac{k+\alpha}{m+\beta}\right)^2=$$

$$=\frac{1}{(m+\beta)^2}\left\{m^2\sum_{k=0}^{m+p}\widetilde{p}_{m,k}(x)\cdot\left(\frac{k}{m}\right)^2+\right.$$

$$\left.+2\alpha m\sum_{k=0}^{m+p}\widetilde{p}_{m,k}(x)\frac{k}{m}+\alpha^2\sum_{k=0}^{m+p}\widetilde{p}_{m,k}(x)\right\}=$$

$$=\frac{1}{(m+\beta)^2}\left\{m^2\widetilde{B}_{m,p}(s^2;x)+2\alpha m\widetilde{B}_{m,p}(s;x)+\alpha^2\widetilde{B}_{m,p}(1;x)\right\}$$

But (see ([7]))

$$\widetilde{B}_{m,p}(s^2;x)=\frac{m+p}{m^2}\left\{(m+p)x^2+x(1-x)\right\}$$

Taking into account of the above equalities, we get

$$\begin{aligned}\widetilde{S}_{m,p}^{(\alpha,\beta)}(s^2;x) &= \frac{1}{(m+\beta)^2}\left\{(m+p)^2x^2+(m+p)x(1-x)+\right.\\ &+ \left.2\alpha m\cdot\frac{m+p}{m+\beta}x+2\alpha^2\cdot\frac{m}{m+\beta}+\alpha^2\right\}=\\ &= \frac{1}{(m+\beta)^2}\left\{(m+p)^2x^2+(m+p)x(1-x)+\right.\\ &+ \left.2\frac{\alpha m(m+p)}{m+\beta}x+\frac{\alpha^2(3m+\beta)}{m+\beta}\right\}\end{aligned}$$

i.e. (2.3) holds and the proof ends. □

Lemma 2.3. *The operators* (1.3) *verify*

$$\widetilde{S}_{m,p}^{(\alpha,\beta)}((e_1-x)^2;x)=\frac{(p-\beta)^2}{(m+\beta)^2}x^2+\frac{m+p}{(m+\beta)^2}x(1-x)+$$

$$+\frac{2\alpha(mp-2m\beta-\beta^2)}{(m+\beta)^3}x+\frac{\alpha^2(3m+\beta)}{(m+\beta)^3} \tag{2.4}$$

Proof. The linearity of $\widetilde{S}_{m,p}^{(\alpha,\beta)}$ (see Lemma 2.1) leads us to

$$\begin{aligned}\widetilde{S}_{m,p}^{(\alpha,\beta)}((e_1-x)^2;x) &= \widetilde{S}_{m,p}^{(\alpha,\beta)}(s^2;x)-2x\widetilde{S}_{m,p}^{(\alpha,\beta)}(s;x)+\\ &+ x^2\widetilde{S}_{m,p}^{(\alpha,\beta)}(1;x)\end{aligned}$$

Applying next Lemma 2.2, we get (2.4). □

We are now ready to establish an important convergence property of the sequence $\left\{\widetilde{S}_{m,p}^{(\alpha,\beta)}f\right\}_{m\in\mathbb{N}}$ contained in

Theorem 2.1. *The sequence* $\left\{\widetilde{S}_{m,p}^{(\alpha,\beta)}f\right\}_{m\in\mathbb{N}}$ *converges to* f*, uniformly on* $[0,1]$*, for any* $f \in C([0,1+p])$.

Proof. Because

$$\lim_{m\to\infty}\left\{\frac{(p-\beta)^2}{(m+\beta)^2}x^2\frac{m+p}{(m+\beta)^2}x(1-x)+\frac{2\alpha(mp-2m\beta-\beta^2)}{(m+\beta)^3}x+\frac{\alpha^2(3m+\beta)}{(m+\beta)^3}\right\}=0$$

uniformly on $[0,1]$, we can apply the well known Bohman-Korovkin Theorem and we arrive to the desired result. □

For evaluating the rate of convergence, we will use the first order modulus of smoothness (see ([1])). Let us to recall the definition of this modulus.

Definition 2.1. Let $f : [a,b] \to \mathbb{R}$ be a real valued function, bounded on $[a,b]$. The first order modulus of smoothness is the function $\omega_1 : [0,b-a] \to [0,+\infty)$, defined for any $\delta \in [0,b-a]$ by

$$\omega_1(f;\delta)=\sup\{|f(x)-f(x')| : x,x' \in [0,b-a],\ |x-x'|\le\delta\} \tag{2.5}$$

It is well known the following result, due to O. Shisha and B. Mond (see([8])).

Theorem 2.2. *Let* $(L_m)_{m\in\mathbb{N}}$*,* $L_m : C([a,b]) \to B([a,b])$ *be a sequence of linear positive operators, reproducing the constant functions. For any* $f \in C([a,b])$*, any* $x \in [a,b]$ *and any* $\delta \in [0,b-a]$*, the following*

$$|(L_mf)(x)-f(x)| \le \left\{1+\delta^{-1}\sqrt{L_m((e_1-x)^2;x)}\right\}\omega_1(\delta) \tag{2.6}$$

holds.

Theorem 2.3. *For any* $f \in C([0,1+p])$ *and any* $x \in [0,1]$ *the Schurer-Stancu operators* (1.3) *verify*

$$\left|\left(\widetilde{S}_{m_1}^{(\alpha,\beta)}f\right)(x)-f(x)\right| \le 2\omega_1\left(\sqrt{\delta_{m,p,\alpha,\beta,x}}\right) \tag{2.7}$$

where:

$$\begin{aligned}\delta_{m,p,\alpha,\beta,x} &= \frac{(p-\beta)^2}{(m+\beta)^2}+\frac{m+p}{(m+\beta)^2}x(1-x)+\\ &+ \frac{2\alpha(mp-2m\beta-\beta^2)}{(m+\beta)^3}x+\frac{\alpha^2(3m+\beta)}{(m+\beta)^2}\end{aligned} \tag{2.8}$$

$$\beta \in \left[0,\sqrt{m^2+mp}\,\right] \tag{2.9}$$

Proof. Applying Theorem 2.2 and Lemma 2.3, follows

$$\left|\left(S_m^{(\alpha,\beta)}f\right)(x)-f(x)\right| \le \left(1+\delta^{-1}\cdot\sqrt{\delta_{m,p,\alpha,\beta,x}}\right)\omega_1(\delta)$$

for any $\delta > 0$. Choosing $\delta = \sqrt{\delta_{m,p,\alpha,\beta,x}}$ in the above inequality we arrive to (2.8) and the proof ends. □

Remark 2.1. In Theorem 2.3 is expressed the order of local approximation of f by $\widetilde{S}_m^{(\alpha,\beta)}f$. For obtaining the order of global approximation, we must take in (2.8) the maximum of $\delta_{m,p,\alpha,\beta,x}$ when $x \in [0,1]$. Clearly, this maximum depends of the relations between α, β, p.

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