STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVIII**, Number 3, September 2003

SOME APPLICATIONS OF AN ASYMPTOTICAL FIXED POINT THEOREMS FOR INTEGRAL EQUATIONS WITH DEVIATING ARGUMENT

T. BARANYAI

Dedicated to Professor Gheorghe Micula at his 60^{th} *anniversary*

Abstract. In this paper we will present an application of an asymptotical fixed point theorem for integral equation with deviating argument.

The following result is well known : ([1], [2])

Theorem 1. *Let the following integral equation with deviating argument:*

$$x(t) = h(t) + \int_a^t f(s, x(g(s)))ds,\ t \in [a, b]. \tag{1}$$

We suppose that:

(a) $h \in C([a, b], [a, b]),\ h(a) = 0$

(b) $g : [a, b] \longrightarrow [a, b],\ a \leq g(t) \leq t \leq b$

(c) $f \in C([a, b] \times \mathbb{R})$

$\exists L_f > 0,\ |f(t, u) - f(t, v)| \leq L_f\,|u - v|$ *for all* $t \in [a, b],\ u, v \in \mathbb{R}$

Then the equation (1) has an unique solution in $C[a, b]$.

In proving of this theorem are apply the contraction principle for the following operator:

$$A : C[a, b] \longrightarrow C[a, b]\,,$$

$$A(x)(t) := h(t) + \int_a^t f(s, x(g(s)))ds,\ t \in [a, b].$$

In the following we prove the existence and the unicity of the solution of the integral equation (1) without using condition (b) for the operator g. In the proof of theorem 1 are use the Bielicki norm, but in the following theorem we use the Cebîşev norm and an asymptotic fixed point principle.

Received by the editors: 29.09.2003.

2000 *Mathematics Subject Classification.* 45D05, 47H10.

Key words and phrases. Banach space, asymptotical fixed point theorems, integral equations with deviating argument.

Let X be a Banach space. We consider the following integral equation:

$$x(t) = h(t) + \int_a^t f(s, x(g(s)))ds, \ t \in [a, b] \tag{2}$$

Theorem 2. *We suppose that:*

(a) $g \in C([a, b], [a, b])$
(b) $h \in C([a, b], [a, b])$, $h(a) = 0$
(c) $f \in C([a, b] \times X, X)$
$\exists L_f > 0$, $\|f(t, u) - f(t, v)\|_X \le L_f \|u - v\|_X$ *for all* $t \in [a, b]$, $u, v \in X$.

Then the equation (2) has an unique solution in $C([a, b], X)$.

Proof. We consider the operator

$$A : C([a, b], X) \longrightarrow C([a, b], X)$$

$$A(x)(t) := h(t) + \int_a^t f(s, x(g(s)))ds,$$

Then the iterates of A are:

$$\begin{aligned} A^2(x)(t) &= h(t) + \int_a^t f(s, A(x)(g(s)))ds, \\ &\dots \\ A^{n+1}(x)(t) &= h(t) + \int_a^t f(s, A^n(x)(g(s)))ds \end{aligned}$$

We have the following estimations ([3]):

$$|A(x)(t) - A(y)(t)| \le L_f \int_a^t |x(g(s)) - y(g(s))| \, ds \le$$

$$\le L_f \|x - y\|_C \frac{t - a}{1!}, \quad \forall t \in [a, b] \quad (\| \, \|_C \text{ is the Cebîşev norm})$$

$$\left|A^2(x)(t) - A^2(y)(t)\right| \le L_f \int_a^t |x(A(s)) - y(A(s))| \, ds \le$$

$$\le L_f \|x - y\|_C \int_a^t \frac{s - a}{1!} ds \le L_f^2 \|x - y\|_C \frac{(t - a)^2}{2!}, \quad \forall t \in [a, b]$$

$$\dots$$

$$\left|A^k(x)(t) - A^k(y)(t)\right| \le L_f^k \|x - y\|_C \frac{(t - a)^k}{k!}, \quad \forall t \in [a, b], \forall k \in \mathbb{N}$$

$$\left\|A^k(x) - A^k(y)\right\| \le \frac{[L_f(b - a)]^k}{k!} \|x - y\|_C \, , \quad \forall k \in \mathbb{N}.$$

So there exists a natural number k such that that:

A^k is contraction with the contraction constant $\alpha = \dfrac{[L_f(b-a)]^k}{k!} < 1$.

Now we apply an asymptotical variant of contraction principle ([2]) and we have that, the integral equation (2) has an unique solution. Q.E.D.

Remarks.

1. When we take $X = \mathbb{R}^m$ we have a result for the following system of integral equations:

$$\begin{aligned} x_1(t) &= h_1(t) + \int_a^t f_1(s, x_1(g(s)), \dots, x_m(g(s))))ds \\ x_2(t) &= h_2(t) + \int_a^t f_2(s, x_1(g(s)), \dots, x_m(g(s))))ds \qquad t \in [a,b] \\ &\dots \\ x_m(t) &= h_m(t) + \int_a^t f_m(s, x_1(g(s)), \dots, x_m(g(s))))ds \end{aligned}$$

2. When $X = l^2(\mathbb{R})$ we have a result for the following infinit sistem of integral equations:

$$x_i(t) = h_i(t) + \int_a^t f_i(s, x_1(g(s)), \dots, x_n(g(s)), \dots)ds, \quad t \in [a,b],\ i \in \mathbb{N}^*.$$

References

[1] I. A. Rus, *Principii şi aplicaţii ale teoriei punctului fix*, Editura Dacia, Cluj-Napoca, 1979.
[2] I. A. Rus, *Generalized contractions and applications*, Cluj University Press, Cluj-Napoca, 2001.
[3] I. A. Rus, *Aplicaţii cu iterate φ-contracţii*, Studia Univ. Babeş-Bolyai, Mathematica, **25**, (1980), fasc.4, 47-50.

Department of Mathematics, Babeş-Bolyai University,
Teacher Training College, Satu Mare
E-mail address: baratun@yahoo.com