GENERAL EXISTENCE RESULTS FOR THE ZEROS OF A COMPACT NONLINEAR OPERATOR DEFINED IN A FUNCTIONAL SPACE

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Dedicated to Professor Gheorghe Micula at his 60^{th} anniversary

Abstract. Let X be a Banach space whose the elements are functions defined on a non-empty set Ω with values in a prehilbertian space H. Let $B := \{x \in X, \ \|x\| \leq 1\}$, $S := \{x \in X, \ \|x\| = 1\}$ and let $f : \overline{B} \to X$ be a compact operator. one shows that if f fulfills on S certain conditions, then the equation $(*)$ $f(x) = 0$ admits solutions. The particular case when Ω is a topological compact space and $X = C(\Omega, \mathbb{R}^n)$ is also considered.

1. Many existence problems in analysis are reduced to an equation of type

$$f(x) = 0, \tag{1}$$

where f is an operator defined between two adequate functional spaces. Generally, the problem (1) is reduced many times to a fixed point problem for the mapping $x \to x + f(x)$, but not always this reducing is adequate.

Through the results concerning directly the equation (1) we mention the one of Miranda [3], which considers the particular case when f maps in a finite dimensional space. The case considered in what follows is much more general.

2. Let Ω be a non-empty arbitrary set, H be a real prehilbertian space and X be a subset of H^Ω; suppose that X is a Banach space endowed with the norm $\|\cdot\|$.

Denote by $\langle | \rangle$ the scalar product of H and define a mapping from $X \times X$ to $\mathbb{R}^\Omega$,

$$(x, y) \to [x \mid y](t) := \langle x(t) \mid y(t) \rangle, \text{ for all } t \in \Omega. \tag{2}$$

Let us consider

$$[x \mid y] > 0 \tag{3}$$

if

$$[x \mid y](t) > 0, \text{ for all } t \in \Omega, \tag{4}$$

for the inequality "<" the convention being the same.

Denote

$$\overline{B} := \{x \in X, \ \|x\| \leq 1\}, \ S := \{x \in X, \ \|x\| = 1\}.$$

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Let $f : \overline{B} \to X$ be a given operator; one can proof the following result.

Theorem 1. *Suppose that:*

i) f is a compact operator;

ii) $[x \mid f(x)] < 0$, for all $x \in S$;

iii) $0 \notin \overline{f(\overline{B})} \backslash f(\overline{B})$.

Then the equation (1) *admits solutions.*

Proof. By means of contradiction suppose that

$$f(x) \neq 0, \ x \in \overline{B}. \tag{5}$$

Then the operator

$$F(x) := \frac{1}{\|f(x)\|} \cdot f(x) \tag{6}$$

is defined on $\overline{B}$ and is continuous; in addition,

$$F(\overline{B}) \subset S. \tag{7}$$

We shall proof that $F(\overline{B})$ is a relatively compact set.

If $y_n \in F(\overline{B})$, then $y_n = F(x_n)$, $x_n \in \overline{B}$, i.e.

$$y_n = \frac{1}{\|f(x_n)\|} \cdot f(x_n).$$

Since $f(\overline{B})$ is relatively compact, it results that $(f(x_n))_n$ contains a convergent subsequence; one can admit that

$$\lim_{n\to\infty} f(x_n) = z \in \overline{f(\overline{B})}. \tag{8}$$

It remains to show that $z \neq 0$ to conclude that $(y_n)_n$ given by (7) is convergent. But

$$0 \in \overline{f(\overline{B})} \tag{9}$$

implies by (5)

$$0 \in \overline{f(\overline{B})} \backslash f(\overline{B}),$$

which is not true.

Hence, F fulfills the hypotheses of Schauder's fixed point theorem and so it will admit a fixed point which, by (6) will belong to S.

By

$$x = F(x), \ x \in S$$

it results

$$x \cdot \|f(x)\| = f(x), \ x \in S, \tag{10}$$

therefore

$$[x \mid x]^2 \cdot \|f(x)\| = [x \mid f(x)]. \tag{11}$$

But

$$[x \mid x]^2 \geq 0,$$

which contradicts hypothesis ii).

The theorem is proved. □

Remark 1. *Hypothesis iii) can be replaced with a formulation of "aprioric estimate" type, i.e.*

ii) if $0 \in \overline{f\left(\overline{B}\right)}$, *then* $0 \in f\left(\overline{B}\right)$.

Remark 2. *The importance of the result is the fact that one doesn't suppose any link between the topologies of* H *and* X *and the special properties for the applications* $x : \Omega \to X$, *too.*

Remark 3. *Hypothesis iii) is useless if* f *is a closed operator or if* $\dim X < \infty$.

3. In this section we consider the case

$$X = C\left(\Omega, \mathbb{R}^n\right) := \left\{x : \Omega \to \mathbb{R}^n,\ x \text{ continuous}\right\}.$$

Suppose that Ω is a compact topological space and consider in X the norm

$$\|x\| := \sup_{t\in\Omega} |x(t)|,$$

where the norm in $\mathbb{R}^n$ is given by

$$|x| = \max_{1\le i\le n} \left\{|x_i|\right\},\ x = (x_i)_{i\in\overline{1,n}} \in \mathbb{R}^n.$$

Obviously, the result contained in Theorem 1 yields, but in this case one can replace hypothesis ii) with another weaker one. To this aim, set

$$S_i^+ \ :\ = \left\{x \in \overline{B},\ x(t) = (x_j(t))_{j\in\overline{1,n}},\ x_j(t) \equiv 1\right\}$$
$$S_i^- \ :\ = \left\{x \in \overline{B},\ x(t) = (x_j(t))_{j\in\overline{1,n}},\ x_j(t) \equiv -1\right\}.$$

Clearly,

$$\cup_{i=1}^n \left(S_i^+ \cup S_i^-\right) \subset S.$$

Theorem 2. *Suppose that:*

i) $f = (f_i)_{i\in\overline{1,n}} : \overline{B} \to X$ *is a compact operator;*

ii) $\begin{cases} (f_i(x))(t) \le 0,\ x \in S_i^+,\ t \in \Omega,\ i \in \overline{1,n} \\ (f_i(x))(t) \ge 0,\ x \in S_i^-,\ t \in \Omega,\ i \in \overline{1,n} \end{cases}$;

iii) $0 \notin \overline{f\left(\overline{B}\right)} \backslash f\left(\overline{B}\right)$.

Then the equation (1) *admits solutions.*

Proof. As in Theorem 1, if (5) holds, then by using again the operator F, one can deduce similarly the relation (10).

One gets

$$(x \in S) \Longleftrightarrow \left(\sup_{t\in\Omega} |x(t)| = 1\right) \Longleftrightarrow \left((\exists)\ \ t_0 \in \Omega,\ \ |x(t_0)| = 1\right).$$

Hence, since $x \in S$,

$$(\exists)\ \ t_0 \in \Omega,\ \ (\exists)\ \ i \in \overline{1,n},\ |x_i(t_0)| = 1. \tag{12}$$

Suppose firstly that $x_i(t_0) = 1$; by (5) it follows

$$x_i(t_0) \cdot \|f(x)\| = (f_i(x))(t_0),$$

therefore

$$f_i(x)(t_0) > 0.$$

By starting from the fixed point $x = (x_i)_{i\in\overline{1,n}}$ we build $\widetilde{x} : \Omega \to \mathbb{R}^n$ by setting

$$\widetilde{x}(t) = (x_1(t), ..., x_{i-1}(t), 1, x_{i+1}(t), ..., x_n(t)).$$

Obviously,

$$\widetilde{x} \in S_i^+$$

and so, by hypotheses,

$$(f_i(\widetilde{x}))(t) \leq 0,\ t \in \Omega.$$

Since

$$\widetilde{x}(t_0) = x(t_0)$$

and (14) one obtains

$$(f_i(x))(t_0) \leq 0,$$

which contradicts (13). □

References

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