

A COMPARISON OF TWO SYSTEMS DESCRIBING ELECTROMAGNETIC TWO-BODY PROBLEM

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Dedicated to Professor Gheorghe Micula at his 60th anniversary

Abstract. Two systems of equations of motion describing two charged particles in the framework of classical electrodynamics are compared.

1. Introduction

The main purpose of the present paper is to compare two systems equations of motion describing the two-body problem of classical electrodynamics [1], [2] (cf. also [3], [4]). One can see at the end of the paper that even a small difference in the right-hand sides of the equations generates various solutions and completely different physical conclusions for the two-body system.

In 1940 [5] J. L. Synge proposes equations of motion describing the behaviour of two charged particles. His derivations are based on the relativistic form of the pondermotive Lorentz force given by W. Pauli [6] by means of Lienard-Wiechert retarded potentials. J. L. Synge formulates the problem in the Minkowski space, that is, in the framework of the special theory of relativity. Consequently the finite velocity of the propagation of interaction generates delays which are, although implicitly, in the arguments of the unknown velocities of the moving particles in the equations of motion [5]. This does not come as a surprise because the theory of differential equations with retarded argument is formulated about twenty years later (cf. A. D. Myshkis [7]).

In order to overcome this difficulty J. L. Synge [5] builds a sequence of successive approximations such that on every step one has to solve a system of ordinary differential equations. Although there is no a convergence theorem for the successive approximations he proposes some idea for solving of the system. In a recent paper [8], however, we have shown that not only a convergence theorem cannot be proved, but even a sequence of successive approximations could not be constructed in such a way. On the base of the same method [5] J. L. Synge calculates the energy on every step (of successive approximations) and makes a conclusion that the two-body system is not stable (cf. p.139, [5]).

Later in 1963 R. D. Driver [9] recognizes the system obtained in [5] as a functional differential system with delays depending on the unknown trajectories and obtains a correct formulation of the Synge problem even in 1-dimensional case. Since we have taken the same point of view for the type of the delays we are able to compare the systems for 3-dimensional case considered in [1] and [2]-[4].

Prior to begin the main exposition we want to discuss one more difficulty concerning Synge equations. They are 8 in number, while the unknown functions are 6 in number. The problem mentioned is not considered in [1] and related known papers. In [4] (cf. also [2]) we show that the system of equations of motion is equivalent to the one consisting of 6 equations. More precisely, the 4-th and 8-th equation are a consequence of the rest ones.

In the present paper we recall some formulation from [3] and [2] in order to obtain 3-dimensional case of J. L. Synge equations. Here we succeed to simplify the right-hand sides of the equations in more extent than [2]-[4]. We present the equations of motion from [1] using our denotations which makes the comparison of both systems easier. Thus we see that equations from [1] can be turned into our ones (we pretend they are Synge's equations) if the constant k (from [1]) is chosen to be $k = \frac{1}{c^3}$ (c the speed of light). Then it is not surprise that the right-hand sides of equations from [1] (c^3 times larger than ours) generates unstable solutions. At the same time it is shown in [2] that Kepler problem for two charged particles has a circle solution.

2. J. L. Synge's equations of motion

As in [2]-[5] we denote by $x^{(p)} = (x_1^{(p)}(t), x_2^{(p)}(t), x_3^{(p)}(t), x_4^{(p)}(t) = ict)$ ($p = 1, 2$) ($i^2 = -1$) the space-time coordinates of the moving particles, by m_p - their proper masses, by e_p - their charges, c - the speed of light. The coordinates of the velocity vectors are $u^{(p)} = (u_1^{(p)}(t), u_2^{(p)}(t), u_3^{(p)}(t))$ ($p = 1, 2$). The coordinates of the unit tangent vectors to the world-lines are (cf. [2], [3]):

$$\lambda_\alpha^{(p)} = \frac{\gamma_p u_\alpha^{(p)}(t)}{c} = \frac{u_\alpha^{(p)}(t)}{\Delta_p} (\alpha = 1, 2, 3), \lambda_4^{(p)} = i\gamma_p = \frac{ic}{\Delta_p} \quad (1)$$

where $\gamma_p = (1 - \frac{1}{c^2} \sum_{\alpha=1}^3 [u_\alpha^{(p)}(t)]^2)^{-\frac{1}{2}}$, $\Delta_p = (c^2 - \sum_{\alpha=1}^3 [u_\alpha^{(p)}(t)]^2)^{\frac{1}{2}}$. It follows $\gamma_p = c/\Delta_p$.

By $\langle ., . \rangle_4$ we denote the scalar product in the Minkowski space, while by $\langle ., . \rangle$ - the scalar product in 3-dimensional Euclidean subspace. The equations of motion modeling the interaction of two moving charged particles are the following (cf. [5]):

$$m_p \frac{d\lambda_r^{(p)}}{ds_p} = \frac{e_p}{c^2} F_{rn}^{(p)} \lambda_n^{(p)} (r = 1, 2, 3, 4) \quad (2)$$

where the elements of proper time are $ds_p = \frac{c}{\gamma_p} dt = \Delta_p dt$ ($p = 1, 2$). Recall that in (2) there is a summation in n ($n = 1, 2, 3, 4$). The elements $F_{rn}^{(p)}$ of the

electromagnetic tensors are derived by the retarded Lienard-Wiechert potentials $A_r^{(p)} = -\frac{e_p \lambda_r^{(p)}}{\langle \lambda^{(p)}, \xi^{(pq)} \rangle_4}$ ($r = 1, 2, 3, 4$), that is, $F_{rn}^{(p)} = \frac{\partial A_n^{(p)}}{\partial x_r^{(p)}} - \frac{\partial A_r^{(p)}}{\partial x_n^{(p)}}$. By $\xi^{(pq)}$ we denote the isotropic vectors (cf. [2], [5])

$$\xi^{(pq)} = (x_1^{(p)}(t) - x_1^{(q)}(t - \tau_{pq}(t)), x_2^{(p)}(t) - x_2^{(q)}(t - \tau_{pq}(t)), x_3^{(p)}(t) - x_3^{(q)}(t - \tau_{pq}(t)), ic\tau_{pq}(t)),$$

where $\langle \xi^{(p,q)}, \xi^{(p,q)} \rangle_4 = 0$ or

$$\tau_{pq}(t) = \frac{1}{c} \left(\sum_{\beta=1}^3 [x_\beta^{(p)}(t) - x_\beta^{(q)}(t - \tau_{pq}(t))]^2 \right)^{\frac{1}{2}}, \quad ((pq) = (12), (21)). \quad (3_{pq})$$

Calculating $F_{rn}^{(p)}$ as in [5] we write equations from (2) in the form:

$$\begin{aligned} \frac{d\lambda_\alpha^{(p)}}{ds_p} &= \frac{Q_p}{c^2} \left\{ \frac{\xi_\alpha^{(pq)} \langle \lambda^{(p)}, \lambda^{(q)} \rangle_4 - \lambda_\alpha^{(q)} \langle \lambda^{(p)}, \xi^{(pq)} \rangle_4}{\langle \lambda^{(q)}, \xi^{(pq)} \rangle_4^3} \left[1 + \left\langle \xi^{(pq)}, \frac{d\lambda^{(q)}}{ds_q} \right\rangle_4 \right] + \right. \\ &+ \left. \frac{1}{\langle \lambda^{(q)}, \xi^{(pq)} \rangle_4^2} \left[\langle \lambda^{(p)}, \xi^{(pq)} \rangle_4 \frac{d\lambda_\alpha^{(q)}}{ds_q} - \left\langle \lambda^{(p)}, \frac{d\lambda^{(q)}}{ds_q} \right\rangle_4 \xi_\alpha^{(pq)} \right] \right\} \quad (\alpha = 1, 2, 3) \quad (4_\alpha) \end{aligned}$$

$$\begin{aligned} \frac{d\lambda_4^{(p)}}{ds_p} &= \frac{Q_p}{c^2} \left\{ \frac{\xi_4^{(pq)} \langle \lambda^{(p)}, \lambda^{(q)} \rangle_4 - \lambda_4^{(q)} \langle \lambda^{(p)}, \xi^{(pq)} \rangle_4}{\langle \lambda^{(q)}, \xi^{(pq)} \rangle_4^3} \left[1 + \left\langle \xi^{(pq)}, \frac{d\lambda^{(q)}}{ds_q} \right\rangle_4 \right] + \right. \\ &+ \left. \frac{1}{\langle \lambda^{(q)}, \xi^{(pq)} \rangle_4^2} \left[\langle \lambda^{(p)}, \xi^{(pq)} \rangle_4 \frac{d\lambda_4^{(q)}}{ds_q} - \left\langle \lambda^{(p)}, \frac{d\lambda^{(q)}}{ds_q} \right\rangle_4 \xi_4^{(pq)} \right] \right\} \quad (4_4) \end{aligned}$$

where $Q_p = e_1 e_2 / m_p$ ($p = 1, 2$). Further on, we have $u^{(q)} \equiv u^{(q)}(t_{pq})$ ($t_{pq} = t - \tau_{pq}$),

$$\lambda^{(q)} = (\gamma_{pq} u_1^{(q)} / c, \gamma_{pq} u_2^{(q)} / c, \gamma_{pq} u_3^{(q)} / c, i\gamma_{pq}) = (u_1^{(q)} / \Delta_{pq}, u_2^{(q)} / \Delta_{pq}, u_3^{(q)} / \Delta_{pq}, ic / \Delta_{pq})$$

$$\text{where } \gamma_{pq} = \left(1 - \frac{1}{c^2} \sum_{\alpha=1}^3 [u_\alpha^{(q)}(t - \tau_{pq}(t))]^2 \right)^{-\frac{1}{2}}, \quad \Delta_{pq} = \left(c^2 - \sum_{\alpha=1}^3 [u_\alpha^{(q)}(t - \tau_{pq}(t))]^2 \right)^{\frac{1}{2}}$$

$$\text{and } \frac{d\lambda_\alpha^{(p)}}{ds_p} = \frac{d(\frac{\gamma_p}{c} u_\alpha^{(p)})}{\frac{c}{\gamma_p} dt} = \frac{d(\frac{u_\alpha^{(p)}}{\Delta_p})}{\Delta_p dt} = \frac{1}{\Delta_p^2} \dot{u}_\alpha^{(p)} + \frac{u_\alpha^{(p)}}{\Delta_p^4} \langle u^{(p)}, \dot{u}^{(p)} \rangle \quad (\alpha = 1, 2, 3)$$

$$\frac{d\lambda_4^{(p)}}{ds_p} = \frac{d(i\gamma_p)}{\frac{c}{\gamma_p} dt} = \frac{icd(\frac{1}{\Delta_p})}{\Delta_p dt} = \frac{ic}{\Delta_p^4} \langle u^{(p)}, \dot{u}^{(p)} \rangle, \text{ where the dot means a differentiation in } t.$$

In order to calculate $\frac{d\lambda^{(q)}}{ds_q}$ we need the derivative $\frac{dt}{dt_{pq}} \equiv D_{pq}$ which should be calculated from the relation

$$t - t_{pq} = \frac{1}{c} \left(\sum_{\alpha=1}^3 [x_\alpha^{(p)}(t) - x_\alpha^{(q)}(t_{pq})]^2 \right)^{\frac{1}{2}} \quad (t_{pq} < t \text{ by assumption}).$$

$$\text{So we have } \frac{dt}{dt_{pq}} - 1 = \frac{\sum_{\alpha=1}^3 [x_{\alpha}^{(p)}(t) - x_{\alpha}^{(q)}(t_{pq})][u_{\alpha}^{(p)}(t) \frac{dt}{dt_{pq}} - u_{\alpha}^{(q)}(t_{pq})]}{c \left(\sum_{\alpha=1}^3 [x_{\alpha}^{(p)}(t) - x_{\alpha}^{(q)}(t_{pq})]^2 \right)^{\frac{1}{2}}}.$$

Since (3_{pq}) has a unique solution (cf. [3]) we obtain $c^2 \tau_{pq} (D_{pq} - 1) = \langle \xi^{(pq)}, u^{(p)} \rangle D_{pq} - \langle \xi^{(pq)}, u^{(q)} \rangle$ and we can solve the above equation with respect to D_{pq} : $D_{pq} = \frac{c^2 \tau_{pq} - \langle \xi^{(pq)}, u^{(q)} \rangle}{c^2 \tau_{pq} - \langle \xi^{(pq)}, u^{(p)} \rangle}$. We have also $\frac{d}{ds_p} = \frac{d}{\Delta_p dt}$. Then $\frac{d}{ds_q} = \frac{1}{\Delta_{pq}} \frac{d}{dt_{pq}} = \frac{1}{\Delta_{pq}} \frac{d}{dt} \frac{dt}{dt_{pq}} = \frac{D_{pq}}{\Delta_{pq}} \frac{d}{dt}$;

$$\frac{d\lambda_{\alpha}^{(q)}}{ds_q} = \frac{D_{pq}}{\Delta_{pq}} \frac{d\lambda_{\alpha}^{(q)}}{dt} = \frac{D_{pq}}{\Delta_{pq}} \frac{d(u_{\alpha}^{(q)} / \Delta_{pq})}{dt} = D_{pq} \left[\dot{u}_{\alpha}^{(q)} \frac{1}{\Delta_{pq}^2} + \frac{u_{\alpha}^{(q)}}{\Delta_{pq}^4} \langle u^{(q)}, \dot{u}^{(q)} \rangle \right]$$

($\alpha = 1, 2, 3$);

$$\frac{d\lambda_4^{(q)}}{ds_q} = \frac{icD_{pq}}{\Delta_{pq}^4} \langle u^{(q)}, \dot{u}^{(q)} \rangle; \langle \lambda^{(p)}, \lambda^{(q)} \rangle_4 = \frac{\langle u^{(p)}, u^{(q)} \rangle - c^2}{\Delta_p \Delta_{pq}};$$

$$\langle \lambda^{(p)}, \xi^{(pq)} \rangle_4 = \frac{\langle u^{(p)}, \xi^{(pq)} \rangle - c^2 \tau_{pq}}{\Delta_p}; \langle \lambda^{(q)}, \xi^{(pq)} \rangle_4 = \frac{\langle u^{(q)}, \xi^{(pq)} \rangle - c^2 \tau_{pq}}{\Delta_{pq}};$$

$$\langle \xi^{(pq)}, \frac{d\lambda^{(q)}}{ds_q} \rangle_4 = D_{pq} \left[\frac{1}{\Delta_{pq}^2} \langle \xi^{(pq)}, \dot{u}^{(q)} \rangle + \frac{\langle \xi^{(pq)}, u^{(q)} \rangle - c^2 \tau_{pq}}{\Delta_{pq}^4} \langle u^{(q)}, \dot{u}^{(q)} \rangle \right];$$

$$\langle \lambda^{(p)}, \frac{d\lambda^{(q)}}{ds_q} \rangle_4 = \frac{D_{pq}}{\Delta_p \Delta_{pq}^2} \left[\langle u^{(p)}, \dot{u}^{(q)} \rangle + \frac{\langle u^{(p)}, u^{(q)} \rangle - c^2}{\Delta_{pq}^2} \langle u^{(q)}, \dot{u}^{(q)} \rangle \right].$$

We note that in the last expressions $\xi^{(pq)}$ is 4-dimensional vector in the left-hand sides, while in the right-hand sides $\xi^{(pq)}$ is 3-dimensional part of the first three coordinates.

Replacing the above expressions in (4.α) and (4.4) and performing some obvious transformations we obtain for $(pq) = (12), (21), \alpha = 1, 2, 3$:

$$\begin{aligned} \frac{1}{\Delta_p} \dot{u}_{\alpha}^{(p)} + \frac{u_{\alpha}^{(p)}}{\Delta_p^3} \langle u^{(p)}, \dot{u}^{(p)} \rangle &= \frac{Q_p}{c^2} \left\{ \frac{[c^2 - \langle u^{(p)}, u^{(q)} \rangle] \xi_{\alpha}^{(pq)} - [c^2 \tau_{pq} - \langle u^{(p)}, \xi^{(pq)} \rangle] u_{\alpha}^{(q)}}{[c^2 \tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle]^3} \right. \\ &\cdot \frac{\Delta_{pq}^4 + D_{pq} [\Delta_{pq}^2 \langle \xi^{(pq)}, \dot{u}^{(q)} \rangle + (\langle \xi^{(pq)}, u^{(q)} \rangle - c^2 \tau_{pq}) \langle u^{(q)}, \dot{u}^{(q)} \rangle]}{\Delta_{pq}^2} + \\ &+ D_{pq} \frac{[\langle u^{(p)}, \xi^{(pq)} \rangle - c^2 \tau_{pq}] [\dot{u}_{\alpha}^{(q)} + u_{\alpha}^{(q)} \langle u^{(q)}, \dot{u}^{(q)} \rangle / \Delta_{pq}^2]}{[c^2 \tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle]^2} \\ &\left. - D_{pq} \frac{[\langle u^{(p)}, \dot{u}^{(q)} \rangle + (\langle u^{(p)}, u^{(q)} \rangle - c^2) \langle u^{(q)}, \dot{u}^{(q)} \rangle / \Delta_{pq}^2] \xi_{\alpha}^{(pq)}}{[c^2 \tau_{pq} - \langle u^{(q)}, \xi_{\alpha}^{(pq)} \rangle]^2} \right\}, \end{aligned} \quad (5_{p\alpha})$$

$$\frac{1}{\Delta_p^3} \langle u^{(p)}, \dot{u}^{(p)} \rangle = \frac{Q_p}{c^2} \left\{ \frac{\langle u^{(p)}, \xi^{(pq)} \rangle - \tau_{pq} \langle u^{(p)}, u^{(q)} \rangle}{[c^2 \tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle]^3} \right\}.$$

$$\begin{aligned}
 & \cdot \left[\Delta_{pq}^2 + D_{pq}(\langle \xi^{(pq)}, \dot{u}^{(q)} \rangle + (\langle \xi^{(pq)}, u^{(q)} \rangle - c^2 \tau_{pq}) \langle u^{(q)}, \dot{u}^{(q)} \rangle / \Delta_{pq}^2) \right] + \quad (5_{p4}) \\
 & + D_{pq} \frac{\langle u^{(p)}, \xi^{(pq)} \rangle \langle u^{(q)}, \dot{u}^{(q)} \rangle / \Delta_{pq}^2 - \tau_{pq} \langle u^{(p)}, \dot{u}^{(q)} \rangle - \tau_{pq} \langle u^{(p)}, u^{(q)} \rangle \langle u^{(q)}, \dot{u}^{(q)} \rangle / \Delta_{pq}^2}{[c^2 \tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle]^2} \Bigg\}.
 \end{aligned}$$

One can prove (as in [4]) that (5_{p4}) is a consequence of (5_{pα}). Indeed, multiplying (5_{pα}) by $u_\alpha^{(p)}$, summing up in α and dividing into c^2 we obtain (5_{p4}). Therefore we can consider a system consisting of the 1st, 2nd, 3rd, 5th, 6th and 7th equations. The last equations form a nonlinear functional differential system of neutral type with respect to the unknown velocities (cf. [10]-[13]). The delays τ_{pq} depend on the unknown trajectories by the relations (3_{pq}).

Now we are able to present (5_{pα}) in a suitable form in order to make further simplifications (Recall that we shall not consider (5_{p4}) because it is a consequence of (5_{pα})):

$$\begin{aligned}
 \dot{u}_\alpha^p + \frac{\langle u^{(p)}, \dot{u}^{(p)} \rangle}{\Delta_p^2} u_\alpha^{(p)} &= \frac{Q_p \Delta_p}{c^2 (c^2 \tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle)^2} \left\{ \left[\frac{c^2 - \langle u^{(p)}, u^{(q)} \rangle}{c^2 \tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle} \xi_\alpha^{(pq)} - \right. \right. \\
 & - \left. \frac{c^2 \tau_{pq} - \langle u^{(p)}, \xi^{(pq)} \rangle}{c^2 \tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle} u_\alpha^{(q)} \right] \left[\Delta_{pq}^2 + D_{pq} \langle \xi^{(pq)}, \dot{u}^{(q)} \rangle + D_{pq} \frac{\langle u^{(q)}, \xi^{(pq)} \rangle - c^2 \tau_{pq} \langle u^{(q)}, \dot{u}^{(q)} \rangle}{\Delta_{pq}^2} \right] + \\
 & + D_{pq} (\langle u^{(p)}, \xi^{(pq)} \rangle - c^2 \tau_{pq}) \dot{u}_\alpha^{(q)} + D_{pq} (\langle u^{(p)}, \xi^{(pq)} \rangle - c^2 \tau_{pq}) \frac{\langle u^{(q)}, \dot{u}^{(q)} \rangle}{\Delta_{pq}^2} u_\alpha^{(q)} - \quad (6_{p\alpha}) \\
 & - D_{pq} \langle u^{(p)}, \dot{u}^{(q)} \rangle \xi_\alpha^{(pq)} + D_{pq} \frac{c^2 - \langle u^{(p)}, u^{(q)} \rangle}{\Delta_{pq}^2} \langle u^{(q)}, \dot{u}^{(q)} \rangle \xi_\alpha^{(pq)} \Bigg\}.
 \end{aligned}$$

Let us recall that if $(pq) = (12)$ then $u^{(1)} = u^{(1)}(t)$ and $u^{(2)} = u^{(2)}(t - \tau_{12})$, while when $(pq) = (21)$, then $u^{(2)} = u^{(2)}(t)$ and $u^{(1)} = u^{(1)}(t - \tau_{21})$. Further on from (6_{pα}) we obtain

$$\begin{aligned}
 \dot{u}_\alpha^p + \frac{\langle u^{(p)}, \dot{u}^{(p)} \rangle}{\Delta_p^2} u_\alpha^{(p)} &= \\
 &= \frac{Q_p \Delta_p}{c^2 (c^2 \tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle)^2} \left\{ \frac{[c^2 - \langle u^{(p)}, u^{(q)} \rangle] [\Delta_{pq}^2 + D_{pq} \langle \xi^{(pq)}, \dot{u}^{(q)} \rangle]}{c^2 \tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle} \xi_\alpha^{(pq)} - \right. \\
 & - D_{pq} \frac{(c^2 - \langle u^{(p)}, u^{(q)} \rangle) \langle u^{(q)}, \dot{u}^{(q)} \rangle}{\Delta_{pq}^2} \xi_\alpha^{(pq)} - \\
 & - \frac{c^2 \tau_{pq} - \langle u^{(p)}, \xi^{(pq)} \rangle}{c^2 \tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle} \left[\Delta_{pq}^2 + D_{pq} \langle \xi^{(pq)}, \dot{u}^{(q)} \rangle + D_{pq} \frac{\langle u^{(q)}, \xi^{(pq)} \rangle - c^2 \tau_{pq} \langle u^{(q)}, \dot{u}^{(q)} \rangle}{\Delta_{pq}^2} \right] u_\alpha^{(q)} + \\
 & + D_{pq} (\langle u^{(p)}, \xi^{(pq)} \rangle - c^2 \tau_{pq}) \dot{u}_\alpha^{(q)} + D_{pq} \frac{(\langle u^{(p)}, \xi^{(pq)} \rangle - c^2 \tau_{pq}) \langle u^{(q)}, \dot{u}^{(q)} \rangle}{\Delta_{pq}^2} u_\alpha^{(q)} - \\
 & \left. - D_{pq} \langle u^{(p)}, \dot{u}^{(q)} \rangle \xi_\alpha^{(pq)} + D_{pq} \frac{(c^2 - \langle u^{(p)}, u^{(q)} \rangle) \langle u^{(q)}, \dot{u}^{(q)} \rangle}{\Delta_{pq}^2} \xi_\alpha^{(pq)} \right\}.
 \end{aligned}$$

Obviously the second summand and the last one cancel each other and after a re-arrangement of the rest summands we have:

$$\begin{aligned}
 \dot{u}_\alpha^p + \frac{\langle u^{(p)}, \dot{u}^{(p)} \rangle}{\Delta_p^2} u_\alpha^{(p)} &= \\
 &= \frac{Q_p \Delta_p}{c^2 (c^2 \tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle)^2} \left\{ \left[\frac{(c^2 - \langle u^{(p)}, u^{(q)} \rangle) (\Delta_{pq}^2 + D_{pq} \langle \xi^{(pq)}, \dot{u}^{(q)} \rangle)}{c^2 \tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle} - D_{pq} \langle u^{(p)}, \dot{u}^{(q)} \rangle \right] \xi_\alpha^{(pq)} + \right. \\
 &+ \left[-\frac{c^2 \tau_{pq} - \langle u^{(p)}, \xi^{(pq)} \rangle}{c^2 \tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle} (\Delta_{pq}^2 + D_{pq} \langle \xi^{(pq)}, \dot{u}^{(q)} \rangle) - \right. \\
 &- \frac{c^2 \tau_{pq} - \langle u^{(p)}, \xi^{(pq)} \rangle}{c^2 \tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle} D_{pq} \frac{\langle u^{(q)}, \xi^{(pq)} \rangle - c^2 \tau_{pq}}{\Delta_{pq}^2} \langle u^{(q)}, \dot{u}^{(q)} \rangle + \\
 &+ \left. D_{pq} \frac{(\langle u^{(p)}, \xi^{(pq)} \rangle - c^2 \tau_{pq}) \langle u^{(q)}, \dot{u}^{(q)} \rangle}{\Delta_{pq}^2} \right] u_\alpha^{(q)} + D_{pq} (\langle u^{(p)}, \xi^{(pq)} \rangle - c^2 \tau_{pq}) \dot{u}_\alpha^{(q)} \Big\}.
 \end{aligned}$$

It is easy to see that the second and third summands before $u_\alpha^{(q)}$ cancel each other so that we obtain the following simplified form of (6 $_{p\alpha}$):

$$\begin{aligned}
 \dot{u}_\alpha^p + \frac{\langle u^{(p)}, \dot{u}^{(p)} \rangle}{\Delta_p^2} u_\alpha^{(p)} &= \frac{Q_p \Delta_p}{c^2 (c^2 \tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle)^2} \cdot \\
 &\cdot \left\{ \left[\frac{(c^2 - \langle u^{(p)}, u^{(q)} \rangle) (\Delta_{pq}^2 + D_{pq} \langle \xi^{(pq)}, \dot{u}^{(q)} \rangle)}{c^2 \tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle} - D_{pq} \langle u^{(p)}, \dot{u}^{(q)} \rangle \right] \xi_\alpha^{(pq)} - \right. \quad (7_{p\alpha}) \\
 &- \left. \frac{(c^2 \tau_{pq} - \langle u^{(p)}, \xi^{(pq)} \rangle) (\Delta_{pq}^2 + D_{pq} \langle \xi^{(pq)}, \dot{u}^{(q)} \rangle)}{c^2 \tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle} u_\alpha^{(q)} + D_{pq} (\langle u^{(p)}, \xi^{(pq)} \rangle - c^2 \tau_{pq}) \dot{u}_\alpha^{(q)} \right\}.
 \end{aligned}$$

In the same way we can obtain more suitable (in view of next section) form of the equations (5 $_{p4}$) (although we proved that (5 $_{p4}$) is a consequence of (5 $_{p\alpha}$):

$$\begin{aligned}
 \frac{1}{\Delta_p^2} \langle u^{(p)}, \dot{u}^{(p)} \rangle &= \frac{Q_p \Delta_p}{c^2} \left\{ \frac{\langle u^{(p)}, \xi^{(pq)} \rangle - \tau_{pq} \langle u^{(p)}, u^{(q)} \rangle}{(c^2 \tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle)^3} (\Delta_{pq}^2 + D_{pq} \langle \xi^{(pq)}, \dot{u}^{(q)} \rangle) - \right. \\
 &- \frac{D_{pq} \langle u^{(q)}, \dot{u}^{(q)} \rangle}{\Delta_{pq}^2} \cdot \frac{\langle u^{(p)}, \xi^{(pq)} \rangle - \tau_{pq} \langle u^{(p)}, u^{(q)} \rangle}{(c^2 \tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle)^2} + \\
 &+ \left. \frac{D_{pq} \langle u^{(q)}, \dot{u}^{(q)} \rangle}{\Delta_{pq}^2} \cdot \frac{\langle u^{(p)}, \xi^{(pq)} \rangle - \tau_{pq} \langle u^{(p)}, u^{(q)} \rangle}{(c^2 \tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle)^2} - \frac{D_{pq} \tau_{pq} \langle u^{(p)}, \dot{u}^{(q)} \rangle}{(c^2 \tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle)^2} \right\}, \text{ i.e.}
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{\Delta_p^2} \langle u^{(p)}, \dot{u}^{(p)} \rangle &= \frac{Q_p \Delta_p}{c^2 (c^2 \tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle)^2} \cdot \quad (7_{p4}) \\
 &\cdot \left[\frac{(\langle u^{(p)}, \xi^{(pq)} \rangle - \tau_{pq} \langle u^{(p)}, u^{(q)} \rangle) (\Delta_{pq}^2 + D_{pq} \langle \xi^{(pq)}, \dot{u}^{(q)} \rangle)}{c^2 \tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle} - D_{pq} \tau_{pq} \langle u^{(p)}, \dot{u}^{(q)} \rangle \right] \\
 &(p = 1, 2).
 \end{aligned}$$

3. Equations of motion from [1]

Now we consider the equations of motion, considered in [1], p.79-80, using his denotations:

$x_k(s)$ - the position of the particle k in R^3 at the instant s ($k = 1, 2$);

e_k - the charge of the particle k ($k = 1, 2$);

m_k - the mass of the particle k ($k = 1, 2$);

$r_i = r_i(t)$ - the delays, which satisfy the equations:

(*) $cr_i(t) = |x_i(t) - x_j(t - r_i)|$ ($j \neq i$);

v_i - the normalized velocities, where $x'_i = cv_i$ for $i = 1, 2$;

$u_i = \frac{x_i(t) - x_j(t - r_i)}{cr_i}$, $\gamma_i = 1 - v_j(t - r_i) \cdot u_i$ ($j \neq i$), where "." indicates the dot of scalar product in R^3 .

REMARK 1: $x_k(s)$ is the restriction of the space-time vector $(x_1^{(k)}(s), x_2^{(k)}(s), x_3^{(k)}(s), x_4^{(k)}(s) = ics)$, while u_i corresponds to the restriction of the isotropic vector $\frac{\xi^{(pq)}}{c\tau_{pq}}$ in 3-dimensional Euclidean subspace of the Minkowski space.

The delays $r_i = r_i(t)$ and the equations (*) correspond to $\tau_{pq} = \tau_{pq}(t)$ and to the equations (3_{pq}) respectively. Finally, we use the notation $u^{(k)} = u^{(k)}(s) = (u_1^{(k)}(s), u_2^{(k)}(s), u_3^{(k)}(s))$ for the velocity vectors, where $u_\alpha^{(k)}(s) = \frac{dx_\alpha^{(k)}(s)}{ds}$ ($\alpha = 1, 2, 3$), $k = 1, 2$ so the normalized vector $v_k(s)$ would be equaled to $\frac{u^{(k)}(s)}{c}$, if we should use the notation for it ($k = 1, 2$).

The equations of motion, given in [1], are the following:

$$v'_i = \frac{e_i(1 - v_i^2)^{1/2}}{m_i c} [E_j + (v_i \cdot E_j)(u_i - v_i) - (v_i \cdot u_i)E_j], \quad (**)$$

where $v_i^2 = |v_i|^2 (= v_i \cdot v_i)$ and

$$E_j = \frac{kce_j}{r_i^2 \gamma_i^3} [u_i - v_j(t - r_i)][1 - v_j^2(t - r_i)] + \frac{kce_j}{r_i \gamma_i^3} u_i \times ([u_i - v_j(t - r_i)] \times v'_j(t - r_i)),$$

where " \times " stands for the cross product in R^3 , " $k > 0$ is a constant depending on the units used", and the denotation $v'_j(t - r_i)$ most probably means a derivative with respect to the argument of $v_j(t - r_i)$, (in [1] there is no explanation). Rewrite the right-hand side of (**) in the form:

$$\frac{e_i(1 - v_i^2)^{1/2}}{m_i c} \{ [1 - (v_i \cdot u_i)] E_j + (v_i \cdot E_j)(u_i - v_i) \}$$

and calculate the vector cross product from E_j :

$$E_j = \frac{kce_j}{r_i^2 \gamma_i^3} [1 - v_j^2(t - r_i)][u_i - v_j(t - r_i)] + \frac{kce_j}{r_i \gamma_i^3} (u_i \cdot v'_j(t - r_i)[u_i - v_j(t - r_i)] - \frac{kce_j}{r_i \gamma_i^2} v'_j(t - r_i).$$

(since $u_i \cdot [u_i - v_j(t - r_i)] = |u_i|^2 - v_j(t - r_i) \cdot u_i = 1 - v_j(t - r_i) \cdot u_i = \gamma_i!$). Consequently the equations (**) are equivalent to the following ones:

$$v'_i = \frac{e_i(1 - v_i^2)^{1/2}}{m_i c} \left\{ [1 - (u_i \cdot v_i)] \left[\frac{kce_j}{r_i^2 \gamma_i^3} [1 - v_j^2(t - r_i)][u_i - v_j(t - r_i)] + \right. \right.$$

$$\begin{aligned}
 & + \frac{kce_j}{r_i\gamma_i^3}(u_i \cdot v'_j(t-r_i))[u_i - v_j(t-r_i)] - \frac{kce_j}{r_i\gamma_i^2}v'_j(t-r_i) \Big] + \\
 & + \left[\frac{kce_j}{r_i^2\gamma_i^3}[1 - v_j^2(t-r_i)][(v_i \cdot u_i) - (v_i \cdot v_j(t-r_i))] + \right. \\
 & + \frac{kce_j}{r_i\gamma_i^3}(u_i \cdot v'_j(t-r_i))[(v_i \cdot u_i) - (v_i \cdot v_j(t-r_i))] - \\
 & \left. - \frac{kce_j}{r_i\gamma_i^2}(v_i \cdot v'_j(t-r_i)) \right] (u_i - v_i) \Big\}. \tag{8}
 \end{aligned}$$

Then we can arrange the symbols, including the vectors $u_i, v_i, v_j(t-r_i)$ and $v'_j(t-r_i)$ respectively and we obtain the equivalent equations:

$$\begin{aligned}
 v'_i = & \frac{ke_i e_j (1 - v_i^2)^{1/2}}{m_i r_i \gamma_i^2} \left\{ u_i \left[\frac{[1 - (u_i \cdot v_i)][1 - v_j^2(t-r_i) + r_i(u_i \cdot v'_j(t-r_i))]}{r_i \gamma_i} + \right. \right. \\
 & + \left. \frac{[1 - v_j^2(t-r_i) + r_i(u_i \cdot v'_j(t-r_i))][(v_i \cdot u_i) - (v_i \cdot v_j(t-r_i))]}{r_i \gamma_i} - (v_i \cdot v'_j(t-r_i)) \right] - \\
 & - v_i \left[\frac{[1 - v_j^2(t-r_i) + r_i(u_i \cdot v'_j(t-r_i))][(v_i \cdot u_i) - (v_i \cdot v_j(t-r_i))]}{r_i \gamma_i} - (v_i \cdot v'_j(t-r_i)) \right] - \\
 & \left. - v_j(t-r_i) \frac{[1 - (u_i \cdot v_i)][1 - v_j^2(t-r_i) + r_i(u_i \cdot v'_j(t-r_i))]}{r_i \gamma_i} - v'_j(t-r_i)[1 - (u_i \cdot v_i)] \right\}
 \end{aligned}$$

and finally one has

$$\begin{aligned}
 v'_i = & \frac{ke_i e_j (1 - v_i^2)^{1/2}}{m_i r_i \gamma_i^2} \left\{ u_i \left[\frac{[1 - (v_i \cdot v_j(t-r_i))][1 - v_j^2(t-r_i) + r_i(u_i \cdot v'_j(t-r_i))]}{r_i \gamma_i} - \right. \right. \\
 & \left. \left. - (v_i \cdot v'_j(t-r_i)) \right] - \right. \\
 & - v_i \left[\frac{[1 - v_j^2(t-r_i) + r_i(u_i \cdot v'_j(t-r_i))][(v_i \cdot u_i) - (v_i \cdot v_j(t-r_i))]}{r_i \gamma_i} - (v_i \cdot v'_j(t-r_i)) \right] - \\
 & \left. - v_j(t-r_i) \frac{[1 - (u_i \cdot v_i)][1 - v_j^2(t-r_i) + r_i(u_i \cdot v'_j(t-r_i))]}{r_i \gamma_i} - v'_j(t-r_i)[1 - (u_i \cdot v_i)] \right\}. \tag{9}
 \end{aligned}$$

To compare both systems we present the equations (9), using the denotations from our previous section II.

We have for $i \equiv p, j \equiv q, r_i \equiv \tau_{pq}$, and in view of Remark 1:

$$v_i = v_i(t) \equiv \frac{u^{(p)}}{c}; \quad (1 - v_i^2)^{1/2} = \left(1 - \left\langle \frac{u^{(p)}}{c}, \frac{u^{(p)}}{c} \right\rangle \right)^{1/2} = \frac{\Delta_p}{c};$$

$$v'_i = \frac{d(u^{(p)}/c)}{dt} = \dot{u}^{(p)}/c \quad (u^{(p)} = u^{(p)}(t));$$

$$\begin{aligned}
 u_i &\equiv \frac{x^{(p)}(t) - x^{(q)}(t - \tau_{pq})}{c\tau_{pq}} = \frac{1}{c\tau_{pq}} \left(\xi_1^{(pq)}, \xi_2^{(pq)}, \xi_3^{(pq)} \right); \\
 v_j(t - r_i) &\equiv u^{(q)}/c; \quad v'_j(t - r_i) = \frac{d(u^{(q)}/c)}{dt_{pq}} = \frac{1}{c} \cdot \frac{du^{(q)}}{dt} \cdot \frac{dt}{dt_{pq}} = \frac{D_{pq}\dot{u}^{(q)}}{c} \\
 (u^{(q)} &= u^{(q)}(t - \tau_{pq}) = u^{(q)}(t_{pq})); \\
 \gamma_i &\equiv 1 - \left\langle \frac{u^{(q)}}{c}, \frac{\xi^{(pq)}}{c\tau_{pq}} \right\rangle = \frac{c^2\tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle}{c^2\tau_{pq}}; \\
 1 - (v_i \cdot v_j(t - r_i)) &= 1 - \left\langle \frac{u^{(p)}}{c}, \frac{u^{(q)}}{c} \right\rangle = \frac{c^2 - \langle u^{(p)}, u^{(q)} \rangle}{c^2}; \\
 \frac{1 - v_j^2(t - r_i) + r_i(u_i \cdot v'_j(t - r_i))}{r_i\gamma_i} &\equiv \frac{1 - \left\langle \frac{u^{(q)}}{c}, \frac{u^{(q)}}{c} \right\rangle + \tau_{pq} \left\langle \frac{\xi^{(pq)}}{c\tau_{pq}}, \frac{D_{pq}\dot{u}^{(q)}}{c} \right\rangle}{\tau_{pq} \left(1 - \left\langle \frac{u^{(q)}}{c}, \frac{\xi^{(pq)}}{c\tau_{pq}} \right\rangle \right)} = \\
 &= \frac{\Delta_{pq}^2 + D_{pq}\langle \xi^{(pq)}, \dot{u}^{(q)} \rangle}{c^2\tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle}; \\
 1 - (u_i \cdot v_i) &\equiv 1 - \left\langle \frac{\xi^{(pq)}}{c\tau_{pq}}, \frac{u^{(p)}}{c} \right\rangle = \frac{c^2\tau_{pq} - \langle u^{(p)}, \xi^{(pq)} \rangle}{c^2\tau_{pq}}; \\
 (v_i \cdot u_i) - (v_i \cdot v_j(t - r_i)) &\equiv \frac{\langle u^{(p)}, \xi^{(pq)} \rangle - \tau_{pq}\langle u^{(p)}, u^{(q)} \rangle}{c^2\tau_{pq}}
 \end{aligned}$$

and replacing in (9) we obtain the following 6 scalar equations:

$$\begin{aligned}
 \frac{1}{c}\dot{u}_\alpha^p &= \frac{ke_p e_q \Delta_p (c^2\tau_{pq})^2}{cm_p \tau_{pq} (c^2\tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle)^2} \cdot \\
 &\cdot \left\{ \frac{\xi_\alpha^{(pq)}}{c\tau_{pq}} \left[\frac{c^2 - \langle u^{(p)}, u^{(q)} \rangle}{c^2} \cdot \frac{\Delta_{pq}^2 + D_{pq}\langle \xi^{(pq)}, \dot{u}^{(q)} \rangle}{c^2\tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle} - \frac{D_{pq}}{c^2} \langle u^{(p)}, \dot{u}^{(q)} \rangle \right] - \right. \\
 &- \frac{u_\alpha^{(p)}}{c} \left[\frac{\Delta_{pq}^2 + D_{pq}\langle \xi^{(pq)}, \dot{u}^{(q)} \rangle}{c^2\tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle} \cdot \frac{\langle u^{(p)}, \xi^{(pq)} \rangle - \tau_{pq}\langle u^{(p)}, u^{(q)} \rangle}{c^2\tau_{pq}} - \frac{D_{pq}}{c^2} \langle u^{(p)}, \dot{u}^{(q)} \rangle \right] - \\
 &- \frac{u_\alpha^{(q)}}{c} \cdot \frac{c^2\tau_{pq} - \langle u^{(p)}, \xi^{(pq)} \rangle}{c^2\tau_{pq}} \cdot \frac{\Delta_{pq}^2 + D_{pq}\langle \xi^{(pq)}, \dot{u}^{(q)} \rangle}{c^2\tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle} + \\
 &\left. + \frac{D_{pq}\dot{u}_\alpha^{(q)}}{c} \cdot \frac{\langle u^{(p)}, \xi^{(pq)} \rangle - c^2\tau_{pq}}{c^2\tau_{pq}} \right\} \quad (\alpha = 1, 2, 3; (pq) = (12), (21)),
 \end{aligned}$$

The above system is obviously equivalent to the following one (with $Q_p = \frac{e_p e_q}{m_p}$):

$$\dot{u}_\alpha^p + \frac{kcQ_p \Delta_p}{(c^2\tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle)^2} \cdot \left[\frac{(\langle u^{(p)}, \xi^{(pq)} \rangle - \tau_{pq}\langle u^{(p)}, u^{(q)} \rangle) (\Delta_{pq}^2 + D_{pq}\langle \xi^{(pq)}, \dot{u}^{(q)} \rangle)}{c^2\tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle} - \right.$$

$$\begin{aligned}
 & -D_{pq}\tau_{pq}\langle u^{(p)}, \dot{u}^{(q)} \rangle \Big] u_{\alpha}^{(p)} = \\
 = & \frac{kcQ_p\Delta_p}{(c^2\tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle)^2} \left\{ \left[\frac{(c^2 - \langle u^{(p)}, u^{(q)} \rangle) (\Delta_{pq}^2 + D_{pq}\langle \xi^{(pq)}, \dot{u}^{(q)} \rangle)}{c^2\tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle} - D_{pq}\langle u^{(p)}, \dot{u}^{(q)} \rangle \right] \xi_{\alpha}^{(pq)} - \right. \\
 & \left. \frac{(c^2\tau_{pq} - \langle u^{(p)}, \xi^{(pq)} \rangle) (\Delta_{pq}^2 + D_{pq}\langle \xi^{(pq)}, \dot{u}^{(q)} \rangle)}{c^2\tau_{pq} - \langle u^{(q)}, \xi^{(pq)} \rangle} u_{\alpha}^{(q)} + \right. \quad (10_{p\alpha}) \\
 & \left. + D_{pq} \left(\langle u^{(p)}, \xi^{(pq)} \rangle - c^2\tau_{pq} \right) \dot{u}_{\alpha}^{(q)} \right\} \quad (\alpha = 1, 2, 3; (pq) = (12), (21)).
 \end{aligned}$$

Conclusion remarks

Our goal is to point out the difference between the system of equations of motion (9) (or equivalently (10_{pα})) from [1] and Synge's equations (7_{pα}), (7_{p4}).

1) equations (9) are obtained under assumption that (7_{p4}) should be identities which is not discussed in [1]. On the other hand in [4] we have already proved that (7_{p4}) is a consequence of (7_{pα}), α = 1, 2, 3. It is not obvious that (7_{p4}) is an identity.

2) the right-hand sides of (10_{pα}) and Synge's equations (7_{pα}) differ each other by the multiplier c^3 which is a consequence from $kc = \frac{1}{c^2}$. But this means that the right-hand sides of (10_{pα}) are c^3 -times larger than the right-hand sides of Synge equations (7_{pα}), that is, they have another dimension. Therefore it is not surprised that they possess only unstable solutions, while (7_{pα}) have a circle solution [2].

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