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# A FUNCTIONAL CHARACTERIZATION OF THE SYMMETRIC-DIFFERENCE OPERATION

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**Abstract**. Let $M$ be a set and $\mathcal{P}(M)$ the family of the subsets of $M$. On $\mathcal{P}(M)$ we consider the set of all binary operations $O(\mathcal{P}(M))$ and on $O(\mathcal{P}(M))$ we define a relation that we call the subordination relation. Then we show that the only group operation on $\mathcal{P}(M)$, subordinate to the union, is the symmetric difference.

## 1. Introduction

Let $M$ be an arbitrary set and $\mathcal{P}(M) = \{A|\ A \subset M\}$, the family of the subsets of $M$. On the set of the binary operations on $\mathcal{P}(M)$ we define the following subordination relation:

If $f, g : \mathcal{P}(M) \times \mathcal{P}(M) \to \mathcal{P}(M)$ are binary operation on $\mathcal{P}(M)$, we say that $f$ *is subordinate to* $g$ or that $g$ *subordinates* $f$, if $f(X, Y) \subset g(X, Y)$ for all $X, Y \in \mathcal{P}(M)$ and we denote $f \leq g$.

Our purpose is to determine those operations that confers to $\mathcal{P}(M)$ a group structure and which subordinate the intersection or are subordinated to the union.

## 2. Main results

For $M$ and $\mathcal{P}(M)$ mentioned above, we denote $O(\mathcal{P}(M))$ the set of all binary operation on the set $\mathcal{P}(M)$:

$$O(\mathcal{P}(M)) = \{f : \mathcal{P}(M) \times \mathcal{P}(M) \to \mathcal{P}(M)|\ f \text{ is a function }\}.$$

**Remark 1.** a) Among the usual operations, let us mention:
– the operation $\emptyset$: $f(X, Y) = \emptyset$, for all $X, Y \in \mathcal{P}(M)$;
– the operation $M$: $f(X, Y) = M$, for all $X, Y \in \mathcal{P}(M)$;
– the intersection ($\cap$): $f(X, Y) = X \cap Y$, for all $X, Y \in \mathcal{P}(M)$;
– the union ($\cup$): $f(X, Y) = X \cup Y$, for all $X, Y \in \mathcal{P}(M)$;
– the difference ($\setminus$): $f(X, Y) = X \setminus Y$, for all $X, Y \in \mathcal{P}(M)$;
– the symmetric difference ($\Delta$):

$$f(X, Y) = X\Delta Y = (X \cup Y) \setminus (X \cap Y) = (X \setminus Y) \cup (Y \setminus X)$$

b) The following subordination relations hold?

$$\emptyset \leq \cap \leq \cup \leq M.$$

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c) For $f, g \in O(\mathcal{P}(M))$ given operators, the operations $\cap, \cup$ and $\Delta$ are defined by:

$$(f \cap g)(X,Y) = f(X,Y) \cap g(X,Y),$$
$$(f \cup g)(X,Y) = f(X,Y) \cup g(X,Y),$$
$$(f \Delta g)(X,Y) = f(X,Y) \Delta g(X,Y),$$

for all $X, Y \in \mathcal{P}(M)$.

**Proposition 1.** *The subordinate relation is an order relation, which determines on $O(\mathcal{P}(M))$ a lattice, where:*

$$\inf\{f,g\} = f \cap g \text{ and } \sup\{f,g\} = f \cup g, \text{ for } f, g \in O(\mathcal{P}(M)).$$

**Proof.** Let $i, f, g, u \in O(\mathcal{P}(M))$.

If $i \leq f$ and $i \leq g$, then $i(X,Y) \subset f(X,Y)$ and $i(X,Y) \subset g(X,Y)$. So $i(X,Y) \subset (f \cap g)(X,Y)$. The maximal operation $i$, which verifies this inclusion is $i = f \cap g$.

If $f \leq u$ and $g \leq u$, then $f(X,Y) \subset u(X,Y)$ and $g(X,Y) \subset u(X,Y)$. So $(f \cup g)(X,Y) \subset u(X,Y)$. The minimal operation $u$, which verifies this inclusion is $u = f \cup g$. □

It is known that the operation $\Delta$ determines on $\mathcal{P}(M)$ a group structure and $\Delta \leq U$. We will show that, if $M$ is a finite set, then this property characterizes the symmetric difference, that is $\Delta$ is the unique group operation on $\mathcal{P}(M)$, subordinated to the union.

**Theorem 1.** *If $M$ is a finite set, then the symmetric difference $\Delta$ is the unique binary operation on $\mathcal{P}(M)$ which is subordinated to the union and which determines on $\mathcal{P}(M)$ a group structure.*

**Proof.** a) If we denote by "$*$" an operation which satisfies the requirements of the theorem, from $\emptyset * \emptyset \subset \emptyset$ we have $\emptyset * \emptyset = \emptyset$. So the only element that could be the unit element is $\emptyset$.

b) We show by induction after $|X|$ that $X * X = \emptyset$ for all $X \in \mathcal{P}(M)$.

For $|X| = 0$ we have $x = \emptyset$ and $\emptyset * \emptyset = \emptyset$.

We suppose $X * X = \emptyset$ for all $X \in \mathcal{P}(M)$ with $|X| \leq n$ and let $A \in \mathcal{P}(M)$ with $|A| = n + 1$.

If $X \subset A$, then $X * A \subset X \cup A = A$, so the translation restricted to $\mathcal{P}(M)$ has values in $\mathcal{P}(M)$. Being an injection, it is a surjection, since $\mathcal{P}(A)$ is finite. Thus, there exists the set $B \subset A$ such that $t_A(B) = A * B = \emptyset$. If we suppose that $B \neq A$, then $|B| \leq n$ and from the induction hypothesis we have $B * B = \emptyset$. From $A * B = B * B$ we have $A = B$, which is a contradiction that shows that $A * A = \emptyset$.

c) Using an induction on $|B| = k$ we show that if $A \cap B = \emptyset$, then $A * B = A \cup B$.

For $k = 0$, $A * \emptyset = A \cup \emptyset = A$ is immediately verified since $\emptyset$ is the unit element.

For $k = 1$, $B = \{x\}$, $x \notin A$. If $A * \{x\} = C \subset A \cup \{x\}$ then $C * \{x\} \subset C \cup \{x\}$, that is: $A * (\{x\} * \{x\}) \subset C \cup \{x\}$ or $A * \emptyset \subset C \cup \{x\}$ or $A \subset C \cup \{x\}$. Since $x \notin A$ it follows that $A \subset C$ and $C \subset A \cup \{x\}$. So, either $C = A$ or $C = A \cup \{x\}$. But $C = A * \{x\} \neq A$, so we finally obtain $C = A \cup \{x\}$.

For $k = n + 1$, let $B = B_n \cup \{y\}$ with $|B_n| = n$. $B_n \cap A = \emptyset$ and $y \notin A$, $y \notin B_n$.

We have

$$A * B = A * (B_n \cup \{y\}) = A * (B_n * \{y\}) = (A * B_n) * \{y\} =$$
$$= (A * B_n) \cup \{y\} = (A \cup B_n) \cup \{y\} = A \cup (B_n \cup \{y\}) = A \cup B$$

d) We show that $X * Y = X \Delta Y = (X \setminus Y) \cup (Y \setminus X)$. Let $X \cap Y = Z$, $X \setminus Z = U$, $Y \setminus Z = V$ where $U, V, Z$ are disjoint.

We have

$$X * Y = (Z \cup U) * (Z \cup V) \overset{c)}{=} (U * Z) * (Z * V) =$$
$$= U * (Z * Z) * V \overset{b)}{=} U * \emptyset * V \overset{a)}{=} U * V \overset{c)}{=} U \cup V$$
$$= (X \setminus Z) \cup (Y \setminus Z) = (X \setminus Y) \cup (Y \setminus X) = X \Delta Y. \quad \square$$

**Theorem 2.** *If $M$ is a finite set, then the unique operation on $\mathcal{P}(M)$ which subordinates the intersection and which determines on $\mathcal{P}(M)$ a group structure is the operation $\overline{\Delta}$ defined by:*

$$f(X, Y) = X \overline{\Delta} Y = \overline{X \Delta Y} = M \setminus (X \Delta Y), \quad X, Y \in \mathcal{P}(M).$$

**Proof.** If we denote by "$\top$" such an operation, then $X \cap Y \subset X \top Y \Leftrightarrow \overline{X \top Y} \subset \overline{X} \cup \overline{Y} \Leftrightarrow \overline{\overline{X} \top \overline{Y}} \subset X \cup Y$.

Let us denote $\overline{\overline{X} \top \overline{Y}} = X * Y$ and show that $(\mathcal{P}(M), *)$ is a group.

The function $c : \mathcal{P}(M) \to \mathcal{P}(M)$, $c(X) = \overline{X} = M \setminus X$ is a bijection and the structure induced from the group $(\mathcal{P}(M), \top)$ is $X * Y = c^{-1}(c(X) \top c(Y)) = \overline{\overline{X} \top \overline{Y}}$.

Using now the previous theorem and the relation $X * Y \subset X \cup Y$ we deduce that $* = \Delta$, so $\overline{\overline{X} \top \overline{Y}} = X \Delta Y$ or, equivalent, $X \top Y = \overline{\overline{X} \Delta \overline{Y}} = \overline{X \Delta Y}$. $\square$

**Remark 2.** The proofs of the theorems have essentially used the fact that the set $M$ is finite. It is an open problem whether the results take place for infinite sets.

## References

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