STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVIII**, Number 2, June 2003

ON SOME UNIVALENCE CONDITIONS IN THE UNIT DISK

VERONICA NECHITA

Abstract. In this paper we obtain by the method of subordination chains an univalence criterion for analytic functions defined in the unit disk, which generalizes a criterion due to D.Răducanu.

1. **Introduction**

We denote by U_r the disk $\{z \in \mathbb{C} : |z| < r\}$, where $0 < r \leq 1$ and by $U = U_1$ the unit disk of the complex plane $\mathbb{C}$.

Let $\mathcal{A}$ denote the class of analytic functions in the unit disk U which satisfy the conditions $f(0) = f'(0) - 1 = 0$.

Let f and F be analytic functions in U. The function f is said to be subordinate to F, written $f \prec F$ or $f(z) \prec F(z)$, if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| \leq 1$, and such that $f(z) = F(w(z))$. If F is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

A function $L(z,t)$, $z \in U$, $t \geq 0$ is a subordination chain if $L(\cdot, t)$ is analytic and univalent in U, for all $t \geq 0$, and $L(z,s) \prec L(z,t)$, when $0 \leq s \leq t < \infty$.

Theorem 1. [1] *Let* $r \in (0,1]$ *and* $L : U_r \times [0,\infty) \to \mathbb{C}$ *be an analytic function in the disk* U_r, *for all* $t \geq 0$, $L(z,t) = a_1(t) z + \ldots$. *If*

(i) $L(z,\cdot)$ *is locally absolutely continuous in* $[0,\infty)$, *locally uniform with respect to* U_r,

(ii) there exists a function $p(z,t)$ *analytic in* U *for all* $t \in [0,\infty)$ *and measurable in* $[0,\infty)$ *for each* $z \in U$, *such that* $\operatorname{Re} p(z,t) > 0$, *for* $z \in U$, $t \in [0,\infty)$, *and*

$$\frac{\partial L(z,t)}{\partial t} = z \frac{\partial L(z,t)}{\partial z} p(z,t),$$

for $z \in U_r$, *and for almost all* $t \in [0,\infty)$,

(iii) $a_1(t) \neq 0$, *for* $t \geq 0$, $\lim\limits_{t\to\infty} |a_1(t)| = \infty$ *and* $\left\{ \dfrac{L(z,t)}{a_1(t)} \right\}_{t \geq 0}$ *is a normal family in* U_r,

then for each $t \geq 0$, $L(z,t)$ *has an analytic and univalent extension in* U.

Received by the editors: 18.09.2002.

2. Main Result

Theorem 2. *Let $f \in \mathcal{A}$ be an analytic function in U of the form $f(z) = z + a_2 z^2 + \ldots$ for all $z \in U$, $\alpha \in \mathbb{C}$, $a \in \mathbb{R}$ such that $\left|\frac{2}{a\alpha} - 1\right| \leq 1$ and $\operatorname{Re}(a\alpha - 1) > 0$. If*

$$\left|\left(\frac{2}{a\alpha} - 1\right)\left[1 - (1 - |z|^a)\frac{zf'(z)}{f(z)}\right] + (1 - |z|^a)\, z\frac{d}{dz}\left[\log\frac{z^{\left(\frac{2}{a}+1\right)}(f'(z))^{\frac{2}{a}}}{(f(z))^{\frac{2}{a}+1}}\right]\right| \leq |z|^a, \tag{1}$$

for all $z \in U$, then f is univalent in U.

Proof. Let $L : U \times [0, \infty) \to \mathbb{C}$ be the function

$$L(z,t) := \left[f\left(e^{-t}z\right)\right]^{1-\alpha}\left[f\left(e^{-t}z\right) + \frac{\left(e^{at} - 1\right)e^{-t}zf'\left(e^{-t}z\right)}{1 - \left(e^{at} - 1\right)\left(\frac{e^{-t}zf'\left(e^{-t}z\right)}{f\left(e^{-t}z\right)} - 1\right)}\right]^{\alpha}. \tag{2}$$

Because $f(z) \neq 0$ for all $z \in U \setminus \{0\}$, the function

$$f_1(z,t) := \frac{e^{-t}zf'\left(e^{-t}z\right)}{f\left(e^{-t}z\right)} = 1 + \ldots$$

is analytic in U. Hence, the function

$$f_2(z,t) := \frac{e^{-t}zf'\left(e^{-t}z\right)}{f\left(e^{-t}z\right)} - 1 = a_2 e^{-t}z + \ldots$$

is analytic in U.

It follows from

$$f_3(z,t) := 1 + \frac{\left(e^{at} - 1\right)f_1(z,t)}{1 - \left(e^{at} - 1\right)f_2(z,t)} = e^{at} + \ldots$$

that there exists an $r \in (0,1]$ such that f_3 is analytic in U_r and $f_3(z,t) \neq 0$, for all $z \in U_r$, $t \in [0,\infty)$.

We choose an analytic branch in U_r of the function

$$f_4(z,t) := [f_3(z,t)]^{\alpha} = e^{a\alpha t} + \ldots.$$

We have that

$$L(z,t) = \left[f\left(e^{-t}z\right)\right]^{1-\alpha}\left[f\left(e^{-t}z\right) + \frac{\left(e^{at} - 1\right)e^{-t}zf'\left(e^{-t}z\right)}{1 - \left(e^{at} - 1\right)\left(\frac{e^{-t}zf'\left(e^{-t}z\right)}{f\left(e^{-t}z\right)} - 1\right)}\right]^{\alpha} \tag{3}$$

$$= f\left(e^{-t}z\right)[f_4(z,t)]^{\alpha} = e^{(a\alpha-1)t} + \ldots$$

is an analytic function in U_r.

From (3) we have $L(z,t) = a_1(t)\, z + \ldots$, where

$$a_1(t) = e^{(a\alpha-1)t},$$

$a_1(t) \neq 0$, for all $t \in [0,\infty)$ and $\lim_{t\to\infty}|a_1(t)| = \lim_{t\to\infty} e^{t\operatorname{Re}(a\alpha-1)} = \infty$.

From (2), by a simple calculation, we obtain

$$\frac{\partial L(z,t)}{\partial t} =$$

$$= e^{-t}zf'\left(e^{-t}z\right)\left[f\left(e^{-t}z\right)\right]^{-\alpha}\left[f\left(e^{-t}z\right)+\frac{\left(e^{at}-1\right)e^{-t}zf'\left(e^{-t}z\right)}{1-\left(e^{at}-1\right)\left(\frac{e^{-t}zf'\left(e^{-t}z\right)}{f\left(e^{-t}z\right)}-1\right)}\right]^{\alpha}\cdot$$

$$\cdot\left\{-1+\alpha\frac{a+\left(e^{at}-1\right)\left[-1+\frac{e^{-t}zf'\left(e^{-t}z\right)}{f\left(e^{-t}z\right)}+\frac{e^{-t}zf''\left(e^{-t}z\right)}{f'\left(e^{-t}z\right)}\right]}{1-\left(e^{at}-1\right)\left(\frac{e^{-t}zf'\left(e^{-t}z\right)}{f\left(e^{-t}z\right)}-1\right)}\right\} \tag{4}$$

$$\frac{\partial L(z,t)}{\partial z}=$$

$$= e^{-t}zf'\left(e^{-t}z\right)\left[f\left(e^{-t}z\right)\right]^{-\alpha}\left[f\left(e^{-t}z\right)+\frac{\left(e^{at}-1\right)e^{-t}zf'\left(e^{-t}z\right)}{1-\left(e^{at}-1\right)\left(\frac{e^{-t}zf'\left(e^{-t}z\right)}{f\left(e^{-t}z\right)}-1\right)}\right]^{\alpha}\cdot$$

$$\cdot\left\{1-\alpha\frac{\left(e^{at}-1\right)\left[-1+\frac{e^{-t}zf'\left(e^{-t}z\right)}{f\left(e^{-t}z\right)}+\frac{e^{-t}zf''\left(e^{-t}z\right)}{f'\left(e^{-t}z\right)}\right]}{1-\left(e^{at}-1\right)\left(\frac{e^{-t}zf'\left(e^{-t}z\right)}{f\left(e^{-t}z\right)}-1\right)}\right\} \tag{5}$$

We observe that $\left|\frac{\partial L(z,t)}{\partial t}\right|$ is bounded on $[0,T]$, for any $T>0$ fixed and $z\in U_r$. Therefore, the function L is locally absolutely continuous in $[0,\infty)$, locally uniform with respect to U_r. We also have $\left|\frac{L(z,t)}{a_1(t)}\right|\le k$, for all $z\in U_r$ and $t\in[0,\infty)$. Then, by Montel's Theorem, $\left\{\frac{L(z,t)}{a_1(t)}\right\}_{t\in[0,\infty)}$ is a normal family in U_r.

Let $p: U_r\times[0,\infty)\to\mathbb{C}$ be the function defined by

$$p(z,t)=\frac{\frac{\partial L(z,t)}{\partial t}}{z\frac{\partial L(z,t)}{\partial z}}$$

If the function

$$w(z,t)=\frac{1-p(z,t)}{1+p(z,t)}=\frac{z\frac{\partial L(z,t)}{\partial z}-\frac{\partial L(z,t)}{\partial t}}{z\frac{\partial L(z,t)}{\partial z}+\frac{\partial L(z,t)}{\partial t}}. \tag{6}$$

is analytic in $U\times[0,\infty)$ and $|w(z,t)|<1$, for all $z\in U$ and $t\ge 0$, then p has an analytic extension with positive real part in U, for all $t\ge 0$.

From (4), (5) and (6) we obtain

$$w(z,t)=\left(\frac{2}{a\alpha}-1\right)\left[e^{at}-\left(e^{at}-1\right)\frac{e^{-t}zf'\left(e^{-t}z\right)}{f\left(e^{-t}z\right)}\right]+$$
$$+\left(e^{at}-1\right)\left[\frac{2}{a}+1-\left(\frac{2}{a}+1\right)\frac{e^{-t}zf'\left(e^{-t}z\right)}{f\left(e^{-t}z\right)}+\frac{2}{a}\frac{e^{-t}zf''\left(e^{-t}z\right)}{f'\left(e^{-t}z\right)}\right].$$

We have $|w(z,0)| = \left|\frac{2}{a\alpha} - 1\right| \leq 1$ for all $z \in U$, with a and α in the conditions of the theorem. For $t > 0$, $|w(z,t)| < \max_{|z|=1} |w(z,t)| = \left|w\left(e^{i\theta}, t\right)\right|$, where $\theta \in \mathbb{R}$, so we have to proove that $\left|w\left(e^{i\theta}, t\right)\right| \leq 1$.

Consider $u = e^{-t}e^{i\theta}$, then $u \in U$ and $|u| = e^{-t}$. We have

$$\left|w\left(e^{i\theta}, t\right)\right| = \left|\left(\frac{2}{a\alpha} - 1\right)\left[\frac{1}{|u|^a} - \left(\frac{1}{|u|^a} - 1\right)\frac{uf'(u)}{f(u)}\right] + \left(\frac{1}{|u|^a} - 1\right)\cdot\right.$$
$$\left.\cdot\left[\frac{2}{a} + 1 - \left(\frac{2}{a} + 1\right)\frac{uf'(u)}{f(u)} + \frac{2}{a}\frac{uf''(u)}{f'(u)}\right]\right|$$

and from (1) it follows that $\left|w\left(e^{i\theta}, t\right)\right| \leq 1$.

Then, by Theorem 1, the function L is a subordination chain and $L(z,0) = f(z)$ is univalent in U. □

Theorem 3. *Let $f \in \mathcal{A}$ be a locally univalent function in U, $f(z) = z + a_2 z^2 + ...$ for all $z \in U$, $a, \alpha \in \mathbb{C}$ such that $\left|\frac{2}{a\alpha} - 1\right| \leq 1$ and $\mathrm{Re}(a\alpha - 1) > 0$. If*

$$\left|\left(\frac{2}{a\alpha} - 1\right)\left[1 - \left(1 - |z|^a\right)\frac{zf'(z)}{f(z)}\right] + \left(1 - |z|^a\right) z \frac{d}{dz}\left[\log \frac{z^{\left(\frac{2}{a}+1\right)}\left(f'(z)\right)^{\frac{2}{a}}}{\left(f(z)\right)^{\frac{2}{a}+1}}\right]\right| \leq |z|^a,$$

for all $z \in U$, where $\frac{z^{\left(\frac{2}{a}+1\right)}\left(f'(z)\right)^{\frac{2}{a}}}{\left(f(z)\right)^{\frac{2}{a}+1}}$ denotes the analytic branch of the function, then f is univalent in U.

Remark 1. *For $a = 2$ we obtain the univalence condition from* [3]

References

[1] J. Becker, *Über die Lösungstruktur einer Differentialgleichung in der konformen Abbildung*, J.Reine Angew. Math. 285(1976), 66-74;

[2] J.A. Pfalzgraff, *K-Ouasiconformal Extension Criteria in the Disk*, Complex Variales 1(1993), 293-301;

[3] D. Răducanu, *An univalence criterion for functions analytic in the unit disk*, Mathematica, Cluj.

Faculty of Mathematics and Computer Science,
Babeş-Bolyai University, 3400 Cluj-Napoca, Romania