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SPLINE APPROXIMATION FOR SOLVING SYSTEM OF FIRST ORDER DELAY DIFFERENTIAL EQUATIONS

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Abstract. In a previous work, [9], the authors introduced a new technique using a spline function to find an approximate solution for first order delay differential equations. In this presented paper, we develop and modify the lemmas in [9] so that the technique can be extended to work for the case of numerical approximation for solving system of first order delay differential equations. Error estimation and convergence are also considered and tested using numerical examples. The stability of the technique is investigated.

1. Introduction

Consider the system of first order delay differential equations of the form:

$$\begin{aligned} y'(x) &= f_1(x, y(x), z(x), y(g(x))),\ a \le x \le b \\ z'(x) &= f_2(x, y(x), z(x), z(g(x))),\ y(x_0) = y_0, z(x_0) = z_0 \\ y(x) &= \phi(x),\ z(x) = \overline{\phi}(x), x \in [a^*, a] \end{aligned} \tag{1}$$

In recent years many studies were devoted to the problems of approximate solutions of system ordinary as well as delay differential equations by spline functions [2-6] and [8-10]. While in [1] A. Ayad investigated the spline approximation for Fredholm integro differential equations. Also G. Micula and H. Akca [7] have studied the numerical solutions of system of differential equations with deviating argument by spline functions. Our introduced method is a one step method o$(h^{m+\alpha})$ in $y^{(i)}(x)$ and $z^{(i)}(x)$ where $i = 0, 1$. The modulus of continuity of $y'(x)$ and $z'(x)$ is o$(h^{\alpha}), 0 < \alpha \le 1$ and m is an arbitrary positive integer which is equal to the number of iterations used in computing the spline function. Assuming $f_1,\ f_2 \in C([a, b] \times R^3)$ we shall investigate the error estimation and convergence as well as the stability of the method.

2. Description of the spline method

Rewriting the system (1) in the following form:

$$\begin{aligned} y'(x) &= f_1(x, u_1, v_1, u_1^*),\ a \le x \le b \\ z'(x) &= f_2(x, u_1, v_1, v_1^*) \\ y(x_0) &= y_0, z(x_0) = z_0,\ y(x) = \phi(x),\ z(x) = \overline{\phi}(x), x \in [a^*, a] \end{aligned} \tag{2}$$

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The function g is called the delay function and it is assumed to be continuous on the interval $[a, b]$ and satisfies the inequality $a^* \leq g(x) \leq x, x \in [a, b]$ and $\phi, \overline{\phi} \in C[a^*, a]$.

Suppose that $f_1 : [a, b] \times R^3 \to R$ is continuous and satisfies the Lipschitz conditions

$$|f_1(x, u_1, v_1, u_1^*) - f_1(x, u_2, v_2, u_2^*)| \leq L_1\{|u_1 - u_2| + |v_1 - v_2| + |u_1^* - u_2^*|\} \quad (3)$$

and there exist a constant B_1 so that

$$|u_1^* - u_2^*| \leq B_1 |f_1(x, u_1, v_1, u_1^*) - f_1(x, u_2, v_2, u_2^*)| \quad (4)$$

Also Suppose that $f_2 : [a, b] \times R^3 \to R$ is continuous and satisfies the Lipschitz conditions

$$|f_2(x, u_1, v_1, v_1^*) - f_2(x, u_2, v_2, v_2^*)| \leq L_2\{|u_1 - u_2| + |v_1 - v_2| + |v_1^* - v_2^*|\} \quad (5)$$

and there exist a constant B_2 so that

$$|v_1^* - v_2^*| \leq B_2 |f_2(x, u_1, v_1, v_1^*) - f_2(x, u_2, v_2, v_2^*)| \quad (6)$$

$$\forall (x, u_1, v_1, u_1^*), (x, u_2, v_2, u_2^*), (x, u_1, v_1, v_1^*), (x, u_2, v_2, v_2^*) \in ([a, b] \times R^3)$$

These conditions assure the existence of a unique solutions of y and z of system (1).

Let $\triangle$ be a uniform partition of the interval $[a, b]$ defined by the nodes

$\triangle : a = x_0 < x_1 ... < x_k < x_{k+1} ... < x_n = b, x_k = x_0 + kh, h = \frac{b-a}{n} < 1$ and $k = 0(1)n - 1$

we define the spline function approximating the solutions y and z by $S(x)$ and $\overline{S}(x)$ where

$$S(x) = \begin{cases} S_\triangle(x), & a \leq x \leq b \\ \phi(x), & a^* \leq x \leq a \end{cases}$$

$$\overline{S}(x) = \begin{cases} \overline{S}_\triangle(x), & a \leq x \leq b \\ \overline{\phi}(x), & a^* \leq x \leq a \end{cases}$$

Choosing the required positive integer m, we define $S_\triangle(x)$ and $\overline{S}_\triangle(x)$ by:

$$S_\triangle(x) = S_k^{[m]}(x) = S_{k-1}^{[m]}(x_k) + \int_{x_k}^{x} f_1(x, S_k^{[m-1]}(x), \overline{S}_k^{[m-1]}(x), S_k^{[m-1]}(g(x)))dx \quad (7)$$

$$\overline{S_\triangle}(x) = \overline{S}_k^{[m]}(x) = \overline{S}_{k-1}^{[m]}(x_k) + \int_{x_k}^{x} f_2(x, S_k^{[m-1]}(x), \overline{S}_k^{[m-1]}(x), \overline{S}_k^{[m-1]}(g(x)))dx \quad (8)$$

where $S_{-1}^{[m]}(x_0) = y_0, \overline{S}_{-1}^{[m]}(x_0) = z_0, S_{-1}^{[m]}(g(x_0)) = \phi(g(x_0)), \overline{S}_{-1}^{[m]}(g(x_0)) = \overline{\phi}(g(x_0))$ with $S_{k-1}^{[m]}(x_k)$ and $\overline{S}_{k-1}^{[m]}(x_k)$ are the left hand limit of $S_{k-1}^{[m]}(x)$ and $\overline{S}_{k-1}^{[m]}(x)$ as $x \to x_k$ of the segment $S_\triangle(x)$ and $\overline{S}_\triangle\,(x)$ defined on $[x_{k-1},\, x_k]$. In equation (7), (8) we use the following m iterations for $x \in [x_k\ ,x_{k+1}], k = 0(1)n-1$ and $j = 1(1)m$

$$\begin{aligned} S_k^{[j]}(x) &= S_{k-1}^{[m]}(x_k) + \int_{x_k}^{x} f_1(x, S_k^{[j-1]}(x), \overline{S}_k^{[j-1]}(x), S_k^{[j-1]}(g(x)))dx \qquad (9) \\ \overline{S}_k^{[j]}(x) &= \overline{S}_{k-1}^{[m]}(x_k) + \int_{x_k}^{x} f_2(x, S_k^{[j-1]}(x), \overline{S}_k^{[j-1]}(x), \overline{S}_k^{[j-1]}(g(x)))dx \\ S_k^{[0]}(x) &= S_{k-1}^{[m]}(x_k) + M_k\ (x - x_k) \\ \overline{S}_k^{[0]}(x) &= \overline{S}_{k-1}^{[m]}(x_k) + \overline{M}_k\ (x - x_k) \\ \text{where } M_k &= f_1(x_k, S_{k-1}^{[m]}(x_k), \overline{S}_{k-1}^{[m]}(x_k), S_{k-1}^{[m]}(g(x_k))) \text{ and} \\ \overline{M}_k &= f_2(x_k, S_{k-1}^{[m]}(x_k), \overline{S}_{k-1}^{[m]}(x_k), \overline{S}_{k-1}^{[m]}(g(x_k))) \end{aligned}$$

Such $S_\triangle(x)$, $\overline{S}_\triangle\,(x) \in C[a,b] \times R^3$ are exist and unique.

3. **Error estimation and convergence**

To estimate the error, we represent the exact solution as described by the following scheme.

$$\begin{aligned} y^{[0]}(x) &= y(x) = y_k + y^{'}(\zeta_k)(x - x_k) \qquad (10) \\ z^{[0]}(x) &= z(x) = z_k + z^{'}(\eta_k)(x - x_k) \end{aligned}$$

where $\zeta_k, \eta_k \in (x_k, x_{k+1})$, $y(x_k) = y_k$, $z(x_k) = z_k$. For $1 \le j \le m$ we write

$$\begin{aligned} y^{[j]}(x) &= y(x) = y_k + \int_{x_k}^{x} f_1(x, y^{[j-1]}(x), z^{[j-1]}(x), y^{[j-1]}(g(x)))dx \qquad (11) \\ z^{[j]}(x) &= z(x) = z_k + \int_{x_k}^{x} f_2(x, y^{[j-1]}(x), z^{[j-1]}(x), z^{[j-1]}(g(x)))dx \end{aligned}$$

Set $\omega(h) = \max\{\omega(y^{'}, h), \omega(z^{'}, h)\}$ where $\omega(y^{'}, h)$ and $\omega(z^{'},\, h)$ are the modului of continuity for the functions $y^{'}(x)$ and $z'(x)$.

Moreover, we denote to the estimated error of $y(x)$ and $z(x)$ at any point $x \in [a,b]$ by:

$$\begin{aligned} e(x) &= |y(x) - S_\triangle(x)|\,,\ e_k = |y_k - S_\triangle(x_k)\,| \qquad (12) \\ \overline{e}(x) &= \left|z(x) - \overline{S}_\triangle(x)\right|,\ \overline{e}_k = \left|z_k - \overline{S}_\triangle(x_k)\right| \end{aligned}$$

Lemma 3.1. *[1]. Let α and β be non negative real numbers and $\{A_i\}_{i=1}^m$ be a sequence satisfying $A_1 \ge 0,\ A_i \le \alpha + \beta A_{i+1}$ for $i = 1(1)m-1$ then:*

$$A_1 \le \beta^{m-1} A_m + \alpha \sum_{i=0}^{m-2} \beta^i$$

Lemma 3.2. *[1]. Let α and β be non negative real numbers, $\beta \ne 1$ and $\{A_i\}_{i=0}^k$ be a sequence satisfying*
$A_0 \ge 0,\ A_{i+1} \le \alpha + \beta A_i$ for $i = 0(1)k$ then:

$$A_{k+1} \le \beta^{k+1} A_0 + \alpha \frac{[\beta^{k+1} - 1]}{[\beta - 1]}$$

Definition 3.1. [4] for any $x \in [x_k \ ,x_{k+1}], k = 0(1)n-1$and $j = 1(1)m$ we define the operator $T_{kj}(x)$ by:

$$T_{kj}(x) = \left| y^{[m-j]}(x) - S_k^{[m-j]}(x) \right| + \left| z^{[m-j]}(x) - \overline{S}_k^{[m-j]}(x) \right| \tag{13}$$

whose norm is defined by: $\|T_{kj}\| = \max_{x \in [x_k, x_{k+1}]} \{T_{kj}(x)\}$

Lemma 3.3. *For any $x \in [x_k \ ,x_{k+1}], k = 0(1)n-1$ and $j = 1(1)m$, then*

$$\begin{aligned} \|T_{km}\| &\le [1 + h(c_0 + \overline{c}_0)] (e_k + \overline{e}_k) + 2h\omega(h) && (14) \\ \|T_{k1}\| &\le c_1(e_k + \overline{e}_k) + c_2 h^m \omega(h) && (15) \end{aligned}$$

where $c_0 = \frac{L_1}{1-L_1B_1}, \overline{c}_0 = \frac{L_2}{1-L_2B_2}, c_1 = \sum_{i=0}^{m} (c_0 + \overline{c}_0)^i$ and $c_2 = 2(c_0 + \overline{c}_0)^{m-1}$ are constants independent of h.

Proof. Using (3), (4), (5), (6), (9), (10), (11) and (12), it is easy to proof the lemma.

Lemma 3.4. *Let $e(x), \overline{e}(x)$ be defined as in (12), then there exist constants $c_3, c_4, \overline{c}_3, \overline{c}_4$ independent of h such that the following inequalities hold:*

$$\begin{aligned} e(x) &\le (1 + hc_3) e_k + hc_3 \overline{e}_k + c_4 h^{m+1} \omega(h) && (16) \\ \overline{e}(x) &\le h\overline{c}_3 e_k + (1 + h\overline{c}_3) \overline{e}_k + \overline{c}_4 h^{m+1} \omega(h) && (17) \end{aligned}$$

where $c_3 = c_0 c_1$, $c_4 = c_0 c_2, \overline{c}_3 = \overline{c}_0 c_1$ and $\overline{c}_4 = \overline{c}_0 c_2$

Proof. Using (3), (4), (7), (11), (12) and (15) we get:

$$\begin{aligned} e(x) &\le \left| y(x) - S_k^{[m]}(x) \right| \le e_k + c_0 \|T_{k1}\| \int_{x_k}^{x} dx \\ &\le (1 + hc_3) e_k + hc_3 \overline{e}_k + c_4 h^{m+1} \omega(h) \end{aligned}$$

Similarly using (5), (6), (8), (11), (12) and (15), we can proof the other part of the lemma where $c_3 = c_0 c_1$, $c_4 = c_0 c_2, \overline{c}_3 = \overline{c}_0 c_1$ and $\overline{c}_4 = \overline{c}_0 c_2$ are constants independent of h.

Definition 3.2. *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of the same order then we say that $A \le B$*
iff:

(i) both a_{ij} and b_{ij} are non negative
(ii) $a_{ij} \le b_{ij} \forall i, j$.

Using matrix notation we let

$$E(x) = [e(x) \quad \overline{e}(x)]^T, E_k = [e_k \quad \overline{e}_k]^T \text{ and } C = [c_4 \quad \overline{c}_4]^T$$

where T stands for the transpose, then from lemma 3.4, we write

$$E(x) \le (I + hA) E_k + Ch^{m+1} \omega(h) \tag{18}$$

where I is the unit matrix of order 2 and $A = \begin{pmatrix} c_3 & c_3 \\ \overline{c}_3 & \overline{c}_3 \end{pmatrix}$.

Definition 3.3. *Let $T = [T_{i,j}]$ be an $m \times n$ matrix, then we define*

$$\|T\| = \max_i \sum_{j=0}^{n} |t_{i,j}|.$$

Using this definition the inequality (18) yields:

$$\|E(x)\| \le (1 + h\|A\|) \|E_k\| + \|C\| h^{m+1} \omega(h).$$

This inequality holds for $x \in [a,b]$. Setting $x = x_{k+1}$, we obtain:

$$\|E_{k+1}\| \leq (1 + h\|A\|)\|E_k\| + \|C\| h^{m+1}\omega(h).$$

Using lemma 3.2 and noting that $\|E_0\| = 0$,we get:

$$\begin{aligned}
\|E(x)\| &\leq \|C\| h^{m+1}\omega(h)\frac{\left[(1 + h\|A\|)^{k+1} - 1\right]}{1 + h\|A\| - 1} \\
&\leq \frac{\|C\|}{\|A\|}\left[\left(1 + \frac{\|A\|(b-a)}{n}\right)^n - 1\right] h^m\omega(h) \\
&\leq \frac{\|C\|}{\|A\|}\left[e^{(\|A\|(b-a))} - 1\right] h^m\omega(h) \\
&\leq c_5 h^m \omega(h) = o(h^{m+\alpha})
\end{aligned}$$

where $c_5 = \frac{\|C\|}{\|A\|}\left[e^{(\|A\|(b-a))} - 1\right]$ is a constant independent of h. Using definition 3.3, we get:

$$\begin{aligned}
e(x) &\leq c_5 h^m \omega(h) \\
\overline{e}(x) &\leq c_5 h^m \omega(h)
\end{aligned} \tag{19}$$

now we are going to estimate $\left|y^{'}(x) - S^{'}_{\Delta}(x)\right|$. Using (3), (4), (7), (11), (12), (15) and (19), we get:

$$\left|y^{'}(x) - S^{'}_{\Delta}(x)\right| \leq c_6 h^m \omega(h)$$

where $c_6 = c_0\left[2c_1c_5 + c_2\right]$ is a constant independent of h. Similarly using (5), (6), (8), (11), (12), (15) and (19), we get:

$$\left|z^{'}(x) - \overline{S}^{'}_{\Delta}(x)\right| \leq c_7 h^m \omega(h)$$

where $c_7 = \overline{c}_0\left[2c_1c_5 + c_2\right]$ is a constant independent of h.

Thus from above lemma we have arrived to the following theorem:

Theorem 3.1. *Let $y(x)$, $z(x)$ be the exact solutions of the system (1). If $S_{\Delta}(x), \overline{S}_{\Delta}(x)$ given by (7), (8) are the approximate solutions for the problem, $f_1, f_2 \in C\left([a,b] \times R^3\right)$, then the inequalities*

$$\begin{aligned}
\left|y^{(q)}(x) - S^{(q)}_{\Delta}(x)\right| &\leq c_8 h^m \omega(h) \\
\left|z^{(q)}(x) - \overline{S}^{(q)}_{\Delta}(x)\right| &\leq c_9 h^m \omega(h)
\end{aligned}$$

hold for all $x \in [a,b]$ and $q = 0, 1$ where c_8 and c_9 are constants independent of h.

4. **Stability of the method**

To study the stability of the method given by (7), (8) we change $S_\Delta(x)$ to $W_\Delta(x)$ and $\overline{S}_\Delta(x)$ to $\overline{W}_\Delta(x)$ where

$$W_\triangle(x) = W_k^{[m]}(x) = W_{k-1}^{[m]}(x_k) + \int_{x_k}^{x} f_1(x, W_k^{[m-1]}(x), \overline{W}_k^{[m-1]}(x), W_k^{[m-1]}(g(x)))dx \quad (20)$$

$$\overline{W}_\triangle(x) = \overline{W}_k^{[m]}(x) = \overline{W}_{k-1}^{[m]}(x_k) + \int_{x_k}^{x} f_2(x, W_k^{[m-1]}(x), \overline{W}_k^{[m-1]}(x), \overline{W}_k^{[m-1]}(g(x)))dx \quad (21)$$

$W_{-1}^{[m]}(x_0) = y_0^*, \overline{W}_{-1}^{[m]}(x_0) = z_0^*, W_{-1}^{[m]}(g(x_0)) = \phi(g(x_0)), \overline{W}_{-1}^{[m]}(g(x_0)) = \overline{\phi}(g(x_0))$, with $W_{k-1}^{[m]}(x_k)$ and $\overline{W}_{k-1}^{[m]}(x_k)$ are the left hand limit of $W_{k-1}^{[m]}(x)$ and $\overline{W}_{k-1}^{[m]}(x)$ as $x \to x_k$ of the segment of $W_\triangle(x)$ and $\overline{W}_\triangle(x)$ defined on $[x_{k-1}, x_k]$. In equations (20) and (21), we use the following m iterations. For $x \in [x_k, x_{k+1}], k = 0(1)n-1$ and $j = 1(1)m$

$$\begin{aligned}
W_k^{[j]}(x) &= W_{k-1}^{[m]}(x_k) + \int_{x_k}^{x} f_1(x, W_k^{[j-1]}(x), \overline{W}_k^{[j-1]}(x), W_k^{[j-1]}(g(x)))dx \quad (22)\\
\overline{W}_k^{[j]}(x) &= \overline{W}_{k-1}^{[m]}(x_k) + \int_{x_k}^{x} f_2(x, W_k^{[j-1]}(x), \overline{W}_k^{[j-1]}(x), \overline{W}_k^{[j-1]}(g(x)))dx\\
W_k^{[0]}(x) &= W_{k-1}^{[m]}(x_k) + N_k\ (x - x_k)\\
\overline{W}_k^{[0]}(x) &= \overline{W}_{k-1}^{[m]}(x_k) + \overline{N}_k\ (x - x_k)\\
N_k &= f_1(x_k, W_{k-1}^{[m]}(x_k), \overline{W}_{k-1}^{[m]}(x_k), W_{k-1}^{[m]}(g(x_k)))\\
\overline{N}_k &= f_2(x_k, W_{k-1}^{[m]}(x_k), \overline{W}_{k-1}^{[m]}(x_k), \overline{W}_{k-1}^{[m]}(g(x_k)))
\end{aligned}$$

Moreover, we use the following notation.

$$e^*(x) = |S_\triangle(x) - W_\triangle(x)|\,,\ e_k^* = |S_\triangle(x_k) - W_\triangle(x_k)| \quad (23)$$

$$\overline{e}^*(x) = \left|\overline{S}_\triangle(x) - \overline{W}_\triangle(x)\right|,\ \overline{e}_k^* = \left|\overline{S}_\triangle(x_k) - \overline{W}_\triangle(x_k)\right|$$

Definition 4.1. *For any $x \in [x_k, x_{k+1}], k = 0(1)n-1$ and $j = 1(1)m$ we define the operator $T_{kj}^*(x)$ by:*

$$T_{kj}^*(x) = \left|S_k^{[m-j]}(x) - W_k^{[m-j]}(x)\right| + \left|\overline{S}_k^{[m-j]}(x) - \overline{W}_k^{[m-j]}(x)\right| \quad (24)$$

whose norm is defined by $\left\|T_{kj}^*\right\| = \max_{x\in[x_k,x_{k+1}]}\{T_{kj}^*(x)\}$.

Lemma 4.1. *For any $x \in [x_k, x_{k+1}], k = 0(1)n-1$ and $j = 1(1)m$, then*

$$\begin{aligned}
\|T_{km}^*\| &\leq [1 + h(c_0 + \overline{c}_0)]\,(e_k^* + \overline{e}_k^*) \quad (25)\\
\|T_{k1}^*\| &\leq c_1(e_k^* + \overline{e}_k^*) \quad (26)
\end{aligned}$$

where $c_0, \overline{c}_0$ and c_1 are constants defined as in lemma 3.3 **Proof**. *Using (3), (4), (5), (6), (9), (22) and (23) it is easy to prove the above lemma*

Lemma 4.2. *Let $e^*(x), \overline{e}^*(x)$ be defined as in (23), then there exist constants $c_3, \overline{c}_3$ independent of h such that the following inequalities hold:*

$$e^*(x) \leq (1 + hc_3)\, e_k^* + hc_3 \overline{e}_k^* \tag{27}$$

$$\overline{e}^*(x) \leq h\overline{c}_3 e_k^* + (1 + h\overline{c}_3)\, e_k^* \tag{28}$$

Proof. *Using (3), (4), (5), (6), (7), (8), (20),(21), (23) and (26) the proof is similar to the proof in lemma 3.4.* On the light of definition 3.2 and matrix notation $E^*(x) = [e^*(x)\ \overline{e}^*(x)]^T$ and $E_k^* = [e_k^*\ \overline{e}_k^*]^T$ then from lemma 4.2, we write

$$E^*(x) \leq (I + hA)\, E_k^* \tag{29}$$

where I and A are matrices defined as in (18) using definition 3.3. The inequality (29) yields:
$\|E^*(x)\| \leq (1 + h\,\|A\|)\,\|E_k^*\|$.

This inequality holds for any $x \in [a,b]$. Setting $x = x_{k+1}$,we get:

$$\left\|E_{k+1}^*\right\| \leq (1 + h\,\|A\|)\,\|E_k^*\|$$

Using lemma 3.2, we obtain:

$$\begin{aligned}\|E^*(x)\| &\leq (1 + h\,\|A\|)^{k+1}\,\|E_0^*\| \\ &\leq \left(1 + \frac{\|A\|\,(b-a)}{n}\right)^n \|E_0^*\| \\ &\leq e^{\|A\|(b-a)}\,\|E_0^*\| \\ &\leq c_{10}\,\|E_0^*\|\end{aligned}$$

where $c_{10} = e^{\|A\|(b-a)}$ is a constant independent of h. Now using definition 3.3, we obtain:

$$\begin{aligned}e^*(x) &\leq c_{10}\,\|E_0^*\| \\ \overline{e}^*(x) &\leq c_{10}\,\|E_0^*\|\end{aligned} \tag{30}$$

To estimate $\left|S_\Delta'(x) - W_\Delta'(x)\right|$ we use (3), (4), (7), (20), (23), (26) and (30), we obtain:

$$\left|S_\Delta'(x) - W_\Delta'(x)\right| \leq c_{11}\,\|E_0^*\|$$

where $c_{11} = 2c_0c_1c_{10}$ is a constant independent of h. Similarly using (5), (6), (8), (21), (23), (26) and (30) we get

$$\left|\overline{S}_\Delta'(x) - \overline{W}_\Delta'(x)\right| \leq c_{12}\,\|E_0^*\|$$

where $c_{12} = 2\overline{c}_0c_1c_{10}$ is a constant independent of h. Thus from above lemma we have arrived to the following theorem

Theorem 4.1. *Let $S_\Delta(x), \overline{S}_\Delta(x)$ given by (7), (8) be the approximate solutions of the problem (1) with the initial conditions $y(x_0) = y_0$, $z(x_0) = z_0$ and let $W_\Delta(x), \overline{W}_\Delta(x)$ given by (20), (21) are the approximate solutions for the same problem with the initial conditions $y^*(x_0) = y_0^*$, $z^*(x_0) = z_0^*$ and $f_1, f_2 \in C\left([a,b] \times R^3\right)$ then the inequalities*

$$\begin{aligned}\left|S_\Delta^{(q)}(x) - W_\Delta^{(q)}(x)\right| &\leq c_{13}\,\|E_0^*\| \\ \left|\overline{S}_\Delta^{(q)}(x) - \overline{W}_\Delta^{(q)}(x)\right| &\leq c_{14}\,\|E_0^*\|\end{aligned}$$

hold for all $x \in [a,b]$ *and* $q = 0,1$ $\|E_0^*\| = \max\{|y_0 - y_0^*|, |z_0 - z_0^*|\}$ *where* c_{13}, c_{14} *are constants independent of* h.

5. **Numerical example**

The method is tested using the following example in the interval $[0,1]$ with step size h=0.1 where $m = 4$ and $m = 5$. To test the stability of the method we do change in the initial condition by adding 0.00001.

Example 5.1. *Consider the system of delay differential equation*

$$\begin{aligned} y^{'}(x) &= y(x) - z(x) + y\left(x/2\right) - e^{x/2} + e^{-x},\ 0 \leq x \leq 1 \\ z^{'}(x) &= -y(x) - z(x) - z\left(x/2\right) + e^{-x/2} + e^{x} \\ y(x) &= e^{x},\ z(x) = e^{-x},\ x \leq 0,\ y(0) = 1,\ z(0) = 1. \\ \textit{The exact solution is } y &= e^{x},\ z = e^{-x}. \end{aligned}$$

Table I

x	m	First Apr.	Absolute error	Second Apr.	Abs diff. bet. the num. sol.
0.1	4	$y = 1.105170911$	7.6×10^{-9}	1.105182139	1.1×10^{-5}
0.1	5	$y = 1.105170918$	2.9×10^{-11}	1.105182147	1.1×10^{-5}
0.2	4	$y = 1.221402377$	3.8×10^{-7}	1.221415306	1.3×10^{-5}
0.2	5	$y = 1.221402778$	2×10^{-8}	1.221415714	1.3×10^{-5}
0.3	4	$y = 1.349851046$	7.8×10^{-6}	1.349866173	1.5×10^{-5}
0.3	5	$y = 1.349859939$	1.1×10^{-6}	1.349875098	1.5×10^{-5}
0.4	4	$y = 1.491771687$	5.3×10^{-5}	1.491789545	1.8×10^{-5}
0.4	5	$y = 1.491836988$	1.2×10^{-5}	1.491854936	1.8×10^{-5}
0.5	4	$y = 1.648505578$	2.2×10^{-4}	1.648526745	2.1×10^{-5}
0.5	5	$y = 1.64878964$	6.8×10^{-5}	1.648811008	2.1×10^{-5}
0.6	4	$y = 1.821472326$	6.5×10^{-4}	1.821497444	2.5×10^{-5}
0.6	5	$y = 1.822380782$	2.6×10^{-4}	1.822406275	2.5×10^{-5}
0.7	4	$y = 2.012179165$	1.6×10^{-3}	2.012208952	3×10^{-5}
0.7	5	$y = 2.014537772$	7.9×10^{-4}	2.014568184	3×10^{-5}

Table II

x	m	First Abr.	Absolute error	Second Apr. Sol.	Abs. diff. bet. the num. sol.
0.1	4	$z = 0.9048374116$	6.4×10^{-9}	0.9048445718	7.2×10^{-6}
0.1	5	$z = 0.9048374182$	1.8×10^{-10}	0.9048445788	7.2×10^{-6}
0.2	4	$z = 0.8187301857$	5.7×10^{-7}	0.8187347665	4.6×10^{-6}
0.2	5	$z = 0.8187307828$	3×10^{-8}	0.8187353697	4.6×10^{-6}
0.3	4	$z = 0.7408112275$	7×10^{-6}	0.740813402	2.2×10^{-6}
0.3	5	$z = 0.7408189118$	6.9×10^{-7}	0.740821138	2.2×10^{-6}
0.4	4	$z = 0.6702800604$	4×10^{-5}	0.6702799171	1.4×10^{-7}
0.4	5	$z = 0.6703255091$	5.5×10^{-6}	0.6703254446	6.6×10^{-8}
0.5	4	$z = 0.6063734706$	1.6×10^{-4}	0.6063710188	2.5×10^{-6}
0.5	5	$z = 0.606555279$	2.5×10^{-5}	0.6065530007	2.3×10^{-6}
0.6	4	$z = 0.5483243125$	4.9×10^{-4}	0.5483194985	4.8×10^{-6}
0.6	5	$z = 0.5488891836$	7.8×10^{-5}	0.548884684	4.5×10^{-6}
0.7	4	$z = 0.4953086589$	1.3×10^{-3}	0.4953013317	7.3×10^{-6}
0.7	5	$z = 0.4967716293$	1.9×10^{-4}	0.4967648351	6.8×10^{-6}

6. Conclusions

A new technique using spline function approximation to numerically solve the system of first order delay differential equation is presented. The convergence and stability are discussed. Also, error analysis and stability are investigated showed in table I where m the number of iterations. Tables I and II show improvements of error analysis and stability. Also, from the sixth column of the tables one can see that the algorithm is stable.

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