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# METHOD ON PARTIAL AVERAGING FOR FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH HUKUHARA'S DERIVATIVE

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## 1. Introduction

In classical system of functional-differential equations it is possible to middle both complete and partial equations. Complete averaging was presented by Bogolubov ([1]).

In this paper, we use a partial middling method in the case of functional-differential inclusions with Hukuhara's derivative, i.e. for inclusions of the form

$$D_h X(t) \in F(t, X_t, ) \tag{1}$$

where $D_h X$ denotes a Hukuhara's derivative ([2]) of a multivalued mapping $X$, $X_t : \Theta \to X_t(\Theta) = X(t + \Theta)$ for $\Theta \in [-r, 0]$, $r > 0$, $F$ is a map from $[0, T] \times C_0$ into $CC(R^n)$, and $C_0$ is a metric space of all continuous mapping $\Phi : [-r, 0] \to \mathrm{Conv}(R^n)$.

The application of this method leads to a reduced form of the initial equations system and is useful in the case when the means of certain functions do not exist.

The results of this paper generalize the results of V. A. Płotnikov ([5]), where the generalized system $\dot{x}(t) \in F(t, x)$ was investigated.

## 2. Notations and definitions

By $\mathrm{Conv}(R^n)$ we will denote the family of all nonempty compact and convex subsets of the real $n$-dimensional Euclidean space $R^n$ with the Hausdorff metric $H$ defined by:

$$H(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B}|a - b|, \sup_{b\in B}\inf_{a\in A}|a - b|\right\}$$

for $A, B \in \mathrm{Conv}(R^n)$, where $|\cdot|$ denotes the Euclidean norm.

It is know that $(\mathrm{Conv}(R^n), H)$ is a complete metric space ([3]). Let $CC(R^n)$ denote the space of all nonempty compact but necessarily convex subsets of $\mathrm{Conv}(R^n)$. By $d$ we will denote the distance between two collections $A, B \in CC(R^n)$ i.e.

$$d(A, B) = \max\left\{\max_{a\in A}\min_{b\in B} H(a, b), \max_{b\in B}\min_{a\in A} H(a, b)\right\} \text{ for } a, b \in \mathrm{Conv}(R^n).$$

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Let us denote by $\rho$ a distance between $A \in CC(R^n)$ and $B \in \mathrm{Conv}(R^n)$ defined by:

$$\rho(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B} H(a,b), \sup_{b\in B}\inf_{a\in A} H(a,b)\right\}$$

Let $X : [0,T] \to \mathrm{Conv}(R^n)$ be a given mapping. Using the definition of the difference in $\mathrm{Conv}(R^n)$ the Hukuhara derivative $D_h X$ ([2]) of $X$ may be introduced in the following way:

$$D_h X(t) = \lim_{h\to 0+} 1/h(X(t+h) - X(t)) = \lim_{h\to 0+} 1/h(X(t) - X(t-h)) \qquad (2)$$

where $X$ is assumed to belong to the class $D$ of all functions such that both differences in (2) are possible.

The mapping $X : [0,T] \to \mathrm{Conv}(R^n)$ will be called Hukuhara differentiable in $[0,T]$ if $D_h X$ exists for every $t \in [0,T]$.

A function $X : [0,T] \to \mathrm{Conv}(R^n)$ is called absolutely continuous if for every positive number $\varepsilon$ there is a positive number $\delta$ such that

$$\sum_{i=1}^{k} H(X(\beta_i), X(\alpha_i)) < \varepsilon \ \text{ whenever } \ \alpha_1 < \beta_1 \le \alpha_2 < \beta_2 \le \ldots \le \alpha_k < \beta_k$$

and $\displaystyle\sum_{i=1}^{k}(\beta_i - \alpha_i) < \delta.$

The Aumann-Hukuhara's integral for multifunction $F : [0,T] \to CC(R^n)$ is a collection $G \in CC(R^n)$ defined by:

$$G = \left\{g \in \mathrm{Conv}(R^n) : g = \int_0^t f(t)dt \text{ for } f(t) \in F(t)\right\}$$

where $f : [0,T] \to \mathrm{Conv}(R^n)$ and integral of $f$ on a set $[0,T]$ is the Hukuhara integral defined in the paper ([2]).

Finally, denote by $C_\alpha$ a metric space of all continuous mapping $V : [-r,\alpha] \to \mathrm{Conv}(R^n)$ where $\alpha \ge 0, r > 0$, with metric $\rho_\alpha$ defined by:

$$\rho_\alpha(V_1,V_2) = \sup_{-r\le t\le \alpha} H(V_1(t),V_2(t)) \text{ for } V_1,V_2 \in C_\alpha.$$

We say that $X$ is a solution of (1) with the initial absolutely continuous multifunctions $\Phi : [-r,0] \to \mathrm{Conv}(R^n)$ if $X$ is an absolutely continuous function from $[-r,T]$ into $\mathrm{Conv}(R^n)$ with the properties:

$X(t) = \Phi(t)$ for $t \in [-r,0]$

and $X$ satisfies the inclusions (1) for a.e. $t \in [0,T]$ .

## 3. The theorem on partial middling

Let $F^i : [0,\infty) \times C_0 \to CC(R^n)$ $(i = 1,2)$ satisfy the following conditions:

1° $F^i(\cdot,U) : [0,\infty) \to CC(R^n)$ is measurable for fixed $U \in C_0$

2° there exists a $M > 0$ such that $d(F^i(t,U),\{0\}) \le M$ for $(t,U) \in [0,\infty) \times C_0$

3° $F^i(t,\cdot) : C_0 \to CC(R^n)$ satisfies for fixed $t \in [0,\infty)$ the Lipschitz condition of the form:

$$d\left(F^i(t,U),F^i(t,V)\right) \le K\rho_0(U,V)$$

where $K > 0$, $U, V \in C_0$

4° there exists a limit

$$\lim_{T\to\infty} \frac{1}{T} d\left(\int_0^T F^1(t,U)dt,\ \int_0^T F^2(t,U)dt\right) = 0$$

uniformly with respect to $U \in C_0$.

In this part we shall study differential inclusions of the form

$$D_h X^1(t) \in \varepsilon F^1(t, X_t^1) \quad \text{for a.e.} \quad t \geq 0 \tag{3}$$

and

$$D_h X^2(t) \in \varepsilon F^2(t, X_t^1) \quad \text{for a.e.} \quad t \geq 0 \tag{4}$$

where $\varepsilon > 0$ is a small parameter.

We shall consider (3) and (4) together with the initial conditions

$$X^1(t) = X^2(t) = \Phi(t) \quad \text{for } t \in [-r, 0] \tag{5}$$

where $\Phi : [-r, 0] \to \mathrm{Conv}R^n$ is a given absolutely continuous multifunction.

In paper ([4]) the following theorem has been proved.

**Theorem 1.** *Let $\delta : [0,T] \to R$ be a non-negative Lebesgue integreable function and let $\Phi \in C_0$ be an absolutely continuous. Suppose that $F : [0,T] \times C_0 \to CC(R^n)$ satisfy the following conditions:*

1) *$F(\cdot, U) : [0,T] \to CC(R^n)$ is measurable for fixed $U \in C_0$*
2) *there exists a $M > 0$ such that $d(F(t,U), \{0\}) \leq M$ for $(t,U) \in [0,T] \times C_0$*
3) *$F(t,\cdot) : C_0 \to CC(R^n)$ satisfies for fixed $t \in [0,T]$ the Lipschitz conditions of the form*

$$d\left(F(t,U), F(t,V)\right) \leq K(t)\rho_0(U,V)$$

*where $K : [0,T] \to R^+$ is a Lebesgue integrable function, $U, V \in C_0$.*

*Furthermore let $Y : [-r,T] \to \mathrm{Conv}(R^n)$ be an absolutely continuous mapping such that*

4) *$Y(t) = \Phi(t)$ for $t \in [-r, 0]$,*
5) *$\rho(D_h Y(t), F(t, Y_t) \leq \delta(t)$ for a.e. $t \in [0,T]$.*

Then there is a solution $X$ of an initial-value problem:

$$\begin{cases} D_h X(t) \in F(t, X_t) \ \text{ for a.e. } t \in [-r, 0], \\ X(t) = \Phi(t) \ \text{ for } t \in [-r, 0] \end{cases}$$

such that $H(X(t), Y(t)) \leq \xi(t)$ for $t \in [0,T]$
and $H(D_h X(t),\ D_h Y(t)) \leq \delta(t) + K(t)\xi(t)$ for a.e. $t \in [0,T]$
where $\xi(t) = \int_0^t \delta(s)\exp[m(t) - m(s)]ds$ and $m(t) = \int_0^t K(r)dr$.

Now we can prove the main result of this paper, where in Theorem 2 by $CC(R^n)$ we will denote the spaces of all nonempty compact and convex subsets of $\mathrm{Conv}(R^n)$.

**Theorem 2.** *Suppose $F^i : [0,\infty) \times C_0 \to CC(R^n)$, $(i = 1, 2, )$ satisfy the conditions 1° – 4°. Then, for each $\eta > 0$ and $T > 0$ there exists a $\varepsilon^0(\eta, T)$ such that for every $\varepsilon \in (0, \varepsilon^0]$ the following conditions are satisfied:*

(i) *for each solution $X^1(\cdot)$ of (3) there exists a solution $X^2(\cdot)$ of (4) such that:*

$$H\left(X^1(t), X^2(t)\right) \le \eta \qquad \textit{for } t \in \left[-r, \frac{T}{\varepsilon}\right] \tag{6}$$

(ii) *for each solution* $X^2(\cdot)$ *of* (4) *there exists a solution* $X^1(\cdot)$ *of* (3) *such that* (6) *holds.*

***Proof.*** Let $X^1(\cdot)$ be a solution of (3) on $[-r, 0]$. In order to prove the theorem we shall consider the solution $X^2(\cdot)$ of the inclusion (4) in such a way that for $t \in [-r, 0]$, $X^1(t) = X^2(t) = \Phi(t)$, hence $H(X^1(t), X^2(t)) = 0 < \eta$. We will prove inequality (6) on the interval $\left[0, \frac{T}{\varepsilon}\right]$. To do this divide the interval $\left[0, \frac{T}{\varepsilon}\right]$ on $m$-subintervals $[t_i, t_{i+1}]$, where $t_i = \frac{iT}{\varepsilon m}$, $i = 0, 1, \ldots, m-1$ and write a solution $X^1(\cdot)$ in the form:

$$\begin{cases} X^1(t) = \Phi(t) \quad \text{for } t \in [-r, 0] \\ X^1(t) = X^1(t_i) + \varepsilon \displaystyle\int_{t_i}^{t} V^1(\tau) d\tau \quad \text{for } t \in [t_i, t_{i+1}] \end{cases} \tag{7}$$

where $V^1(t) \in F^1(t, X^1_t)$.

Let us consider a function $Y^1(\cdot)$ defined by

$$\begin{cases} Y^1(t) = \Phi(t) \quad \text{for } t \in [-r, 0] \\ Y^1(t) = Y^1(t_i) + \varepsilon \displaystyle\int_{t_i}^{t} U^1_{i+1}(\tau) d\tau \quad \text{for } \quad t \in [t_i, t_{i+1}] \end{cases} \tag{8}$$

where $U^1_{i+1}(\cdot)$, $i = 0, 1, \ldots, m-1$ are measurable functions such that $U^1_{i+1}(t) \in F^1(t, Y^1_{t_i})$ and

$$H\left(V^1(t), U^1_{i+1}(t)\right) = \rho\left(V^1(t), F^1(t, Y^1_{t_i})\right) = \min_{U(t) \in F^1(t, Y^1_{t_i})} H\left(V^1(t), U(t)\right).$$

By virtue of (7) for every $t \in [t_i, t_{i+1}]$ we have

$$\begin{aligned} H\left(X^1(t), Y^1(t_i)\right) &= H\left(X^1(t_i) + \varepsilon \int_{t_i}^{t} V^1(\tau) d\tau, Y^1(t_i)\right) \le \\ &\le H\left(X^1(t_i), Y^1(t_i)\right) + \varepsilon M(t - t_i) \le \delta_i + \varepsilon M(t - t_i) \end{aligned}$$

where $\delta_i = H(X^1(t_i), Y^1(t_i))$, $i = 0, 1, \ldots, m-1$.

Furthermore, for $t \in [t_i, t_{i+1}]$, we have

$$\begin{aligned} & H\left(V^1(t), U^1_{i+1}(t)\right) \le d\left(F^1(t, X^1_t), F^1(t, Y^1(t_i))\right) \le \\ & \le K \rho_0(X^1_t, Y^1_{t_i}) \end{aligned} \tag{9}$$

But

$$\begin{aligned} & \rho_0(X^1_t, Y^1_{t_i}) \le \rho_0(X^1_t, X^1_{t_i}) + \rho_0(X^1_{t_i}, Y^1_{t_i}) = \\ & = \sup\nolimits_{-r \le s \le 0} H\left(X^1(t+s), X^1(t_i+s)\right) + \sup\nolimits_{-r \le s \le 0} H\left(X^1(t_i+s), Y^1(t_i+s)\right) \end{aligned}$$

By the definition of $X^1(\cdot)$ and the properties of multifunction $F^1(t, X^1_t)$ we have:

$$\sup_{-r \le s \le 0} H\left(X^1(t+s), X^1(t_i+s)\right) \le \frac{MT}{m} \quad \text{for} \quad t \in [t_i, t_{i+1}]$$

Furthermore by the definition $X^1(\cdot)$ and $Y^1(\cdot)$ and using of (7) and (8), we have

$$\sup_{-r\le s\le 0} H\left(X^1(t_i+s), Y^1(t_i+s)\right) = \sup_{t_i-r\le\tau\le t_i}\left(H(X^1(\tau),(Y^1(\tau)\right) =$$
$$= \sup_{t_i-r\le\tau\le t_i}\left\{H\left(X^1(t_i)+\varepsilon\int_{t_i}^{\tau} V^1(s)ds, Y^1(t_i)+\varepsilon\int_{t_i}^{\tau} U^1_{i+1}(s)ds\right)\right\} \le$$
$$\le \sup_{t_i-r\le\tau\le t_i}\left\{H\left(X^1(t_i),Y^1(t_i)\right)+\varepsilon H\left(\int_{t_i}^{\tau} V^1(s)ds, \int_{t_i}^{\tau} U^1_{i+1}(s)ds\right)\right\} \le$$
$$\le \delta_i + \sup_{t_i-r\le\tau\le t_i}\varepsilon\int_{t_i}^{\tau} d\left(F^1(s,X^1_s),F^1(s,Y^1_{t_i})\right)ds \le$$
$$\le \delta_i + \sup_{t_i-r\le\tau\le t_i}\varepsilon\left\{\int_{t_i}^{\tau}\left[d\left(F^1(s,X^1_s),\{0\}\right)+d\left(F^1(s,Y^1_{t_i}),\{0\}\right)\right]ds\right\} \le \delta_i + 2\varepsilon Mr.$$

Therefore, inequality (9) for $t \in [ti, ti_{i+1}]$ can be written as follows

$$H\left(V^1(t),U^1_{i+1}(t)\right) \le K\left(\frac{MT}{m}+\delta_i+2\varepsilon Mr\right). \tag{10}$$

By virtue of (7), (8) and (10), it follows

$$\delta_i = H(X^1(t_i),Y^1(t_i)) =$$
$$= H\left(X^1(t_{i-1})+\varepsilon\int_{t_{i-1}}^{t_i} V^1(\tau)d\tau, Y^1(t_{i-1})+\varepsilon\int_{t_{i-1}}^{t_i} U^1_i(\tau)d\tau\right) \le$$
$$\le H(X^1(t_{i-1}),Y^1(t_{i-1}))+\varepsilon H\left(\int_{t_{i-1}}^{t_i} V^1(\tau)d\tau, \int_{t_{i-1}}^{t_i} U^1_i(\tau)d\tau\right) \le$$

$$\le \delta_{i-1}+\varepsilon\int_{t_{i-1}}^{t_i} H\left(V^1(\tau),U^1_i(\tau)\right)d\tau \le \delta_{i-1}+\varepsilon K(t_i-t_{i-1})\left(\frac{MT}{m}+\delta_{i-1}+2\varepsilon Mr\right)$$
$$= \delta_{i-1}+\frac{K\cdot T}{m}\left(\frac{MT}{m}+\delta_{i-1}+2\varepsilon Mr\right) = \delta_{i-1}\left(1+\frac{KT}{m}\right)+\frac{KT}{m}\left(\frac{MT}{m}+2\varepsilon Mr\right)$$
$$= \delta_{i-1}\left(1+\frac{a}{m}\right)+\frac{b}{m},$$

where $a = KT$ and $b = KT\left(\frac{MT}{m}+2\varepsilon Mr\right)$.

Hence,

$$\delta_i \le \delta_{i-1}\left(1+\frac{a}{m}\right)+\frac{b}{m} \le \left(1+\frac{a}{m}\right)\left[\delta_{i-2}\left(1+\frac{a}{m}\right)+\frac{b}{m}\right]+\frac{b}{m} \le$$

$$\le \left(1+\frac{a}{m}\right)^i \delta_0 + \left(1+\frac{a}{m}\right)^{i-1}\frac{b}{m}+\ldots\frac{b}{m} =$$

$$= \frac{b}{m}\left(1+\left(1+\frac{a}{m}\right)+\ldots+\left(1+\frac{a}{m}\right)^{i-1}\right) = \frac{b}{a}\left(\left(1+\frac{a}{m}\right)^i - 1\right) \le$$

$$\le \frac{b}{a}(e^a-1) = \frac{M}{m}(2\varepsilon mr+T)(e^{KT}-1),$$

where $i = 0, 1, \ldots, m-1$.

For $t \in [t_i, t_{i+1}]$ we have

$$H(X^1(t), X^1(t_i)) = H\left(X^1(t_i)+\varepsilon\int_{t_i}^t V^1(\tau)d\tau,\ X^1(t_i)\right) \le$$

$$\le \varepsilon H\left(\int_{t_i}^t V^1(\tau)d\tau, \{0\}\right) \le \varepsilon M \cdot \frac{T}{\varepsilon m} = \frac{MT}{m}$$

and $H(Y^1(t), Y^1(t_i)) \le \frac{MT}{m}$.

Hence, we obtain

$$\begin{aligned} &H(X^1(t), Y^1(t)) \le H(X^1(t), X^1(t_i)) + H(X^1(t_i), Y^1(t_i)) \\ &+H(Y^1(t_i), Y^1(t)) \le \frac{2MT}{m} + \frac{M}{m}(2\varepsilon mr+T)(e^{KT}-1) \end{aligned} \tag{11}$$

Now we shall consider the function

$$\begin{cases} Y^2(t) = \Phi(t) \quad \text{for } \ t \in [-r, 0] \\ Y^2(t) = Y^2(t_i) + \varepsilon\displaystyle\int_{t_i}^t U^2_{i+1}(\tau)d\tau \quad \text{for } \ t \in [t_i, t_{i+1}] \end{cases} \tag{12}$$

where $U^2_{i+1}(\cdot), i = 0, 1, 2, \ldots, m-1$, are measurable functions such that $U^2_{i+1}(t) \in F^2(t, Y^1_{t_i})$.

Let us notice that by virtue of condition $4^0$ for each $\eta_1 > 0$ and $T > 0$ there exists a $\varepsilon^0(\eta_1, T) > 0$ such that for every $\varepsilon \le \varepsilon^0$ we have the following inequalities:

$$d\left(\frac{\varepsilon m}{iT}\int_0^{\frac{iT}{\varepsilon m}} F^1(t, Y^1_{t_i})dt, \frac{\varepsilon m}{iT}\int_0^{\frac{iT}{\varepsilon m}} F^2(t, Y^1_{t_i})dt\right) \le \frac{\eta_1}{2i} \tag{13}$$

and

$$d\left(\frac{\varepsilon m}{(i+1)T}\int_0^{\frac{(i+1)T}{\varepsilon m}} F^1(t, Y^1_{t_i})dt, \frac{\varepsilon m}{(i+1)T}\int_0^{\frac{(i+1)T}{\varepsilon m}} F^2(t, Y^1_{t_i})dt\right) \le \frac{\eta_1}{2(i+1)} \tag{14}$$

where $i = 1, 2, \ldots, m-1$. Let us observe that $\frac{(i+1)T}{\varepsilon m} = t_{i+1}$ and $\frac{iT}{\varepsilon m} = t_i$.

By virtue of (13), (14) and the Hausdorff metric condition we have

$$d\left(\int_{t_i}^{t_{i+1}} F^1\left(t, Y_{t_i}^1\right) dt, \int_{t_i}^{t_{i+1}} F^2\left(t, Y_{t_i}^1\right) dt\right) \leq$$

$$\leq d\left(\int_0^{t_{i+1}} F^1\left(t, Y_{t_i}^1\right) dt, \int_0^{t_{i+1}} F^2\left(t, Y_{t_i}^1\right) dt\right) +$$

$$+d\left(\int_0^{t_i} F^1\left(t, Y_{t_i}^1\right) dt, \int_0^{t_i} F^2\left(t, Y_{t_i}^1\right) dt\right) \leq$$

$$\leq \frac{\eta_1}{2i} \cdot \frac{iT}{\varepsilon m} + \frac{\eta_1}{2(i+1)} \cdot \frac{T(i+1)}{\varepsilon m} = \frac{\eta_1 T}{\varepsilon m}.$$

Then

$$H\left(\int_{t_i}^{t_{i+1}} U_{i+1}^1(\tau) d\tau, \int_{t_i}^{t_{i+1}} U_{i+1}^2(\tau) dt\right) \leq \frac{\eta_1 T}{\varepsilon m}$$

and

$$\begin{aligned} &H(Y^1(t_{i+1}), Y^2(t_{i+1})) \leq H(Y^1(t_i), Y^2(t_i)) + \\ &+ \varepsilon H\left(\int_{t_i}^{t_{i+1}} U_{i+1}^1(\tau) d\tau, \int_{ti}^{t_{i+1}} U_{i+1}^2(\tau) dt\right) \leq \\ &\leq H(Y^1(t_i), Y^2(t_i)) + \frac{\eta_1 T}{m} \leq \ldots \leq m \cdot \frac{\eta_1 T}{m} = \eta_1 T, \end{aligned} \tag{15}$$

where $i = 0, 1, \ldots, m-1$.

Using the inequality (15) and the fact that for $t \in [t_i, t_{i+1}]$

$$H(Y^1(t), Y^1(t_i)) \leq \frac{MT}{m} \quad \text{and} \quad H(Y^2(t), Y^2(t_i)) \leq \frac{MT}{m}$$

we have

$$\begin{aligned} &H(Y^1(t), Y^2(t)) \leq H(Y^1(t), Y^1(t_i)) + H(Y^1(t_i), Y^2(t_i)) \\ &+H(Y^2(t_i), Y^2(t)) \leq \frac{2MT}{m} + \eta_1 T \end{aligned} \tag{16}$$

By assumption $3^0$ it follows that

$$d(F^2(t, Y_t^2, )F^2(t, Y_{t_i}^1)) \leq K\rho_0(Y_t^2, Y_{t_i}^1)$$

Similarly, as in the proof of the inequality (9) and making use of the inequality (16) we obtain

$$\rho_0(Y_t^2, Y_{t_i}^1) \leq \rho_0(Y_t^2, Y_{t_i}^2) + \rho_0(Y_{t_i}^2, Y_{t_i}^1)$$

$$\leq \frac{MT}{m} + \frac{2MT}{m} + \eta_1 T = \frac{3MT}{m} + \eta_1 T$$

Hence $d(F^2(t, Y_t^2, )F^2(t, Y_{t_i}^1)) \leq K\left(\frac{3MT}{m} + \eta_1 T\right)$.

By virtue of (12) we have:

$$\rho\left(D_h Y^2(t), \varepsilon F^2(t, Y_t^2)\right) = \rho\left(D_h Y^2(t), \varepsilon F^2\left(t, Y_{t_i}^1\right)\right)$$

$$+d\left(\varepsilon F^2\left(t, Y_{t_i}^1\right), \varepsilon F^2\left(t, Y_t^2\right)\right) \leq K\varepsilon\left(\frac{3MT}{m} + \eta_1 T\right).$$

Now, using existence theorem (see Theorem 1) there exists at least a solution $X^2(\cdot)$ of (4) such that for $t \in [0, T/\varepsilon]$

$$H(Y^2(t), X^2(t)) \leq \int_0^t K\varepsilon \left(\frac{3MT}{m} + \eta_1 T\right) \exp[\varepsilon K(t-s)] ds \leq$$
$$\leq \left(\frac{3MT}{m} + \eta_1 T)(e^{KT} - 1\right).$$

Using the inequalities (11) and (16) it follows

$$H\left(X^1(t), X^2(t)\right) \leq H\left(X^1(t), Y^1(t)\right) + H\left(Y^1(t), Y^2(t)\right) + H\left(Y^2(t), X^2(t)\right)$$

$$\leq \frac{4MT}{m} e^{KT} + 2\varepsilon Mre^{KT} + \eta_1 Te^{KT}.$$

Therefore, choosing $m > \frac{12MTe^{KT}}{\eta}, \eta_1 = \frac{\eta}{3Te^{KT}}$ and $\varepsilon < \frac{\eta}{6Mre^{KT}}$ we get the inequality

$$H(X^1(t), X^2(t)) \leq \eta \quad \text{for } t \in [0, T/\varepsilon].$$

Adopting now the procedure presented above we get the condition (ii). In this way the proof is completed for $t \in \left[-r, \frac{T}{\varepsilon}\right]$.

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