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SHEPARD METHOD - FROM APPROXIMATION TO INTERPOLATION

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Abstract. In this paper we review the local Shepard method that uses Franke-Little weights, in order to give necessary and sufficient conditions when this method yields an interpolation function. Next we give a practical algorithm to solve this problem, which is based on geometric algorithms. This paper is a refinement of the results presented in [5] and [6].

1. The Local Shepard Method

Let consider the bi-dimensional case of two independent variables x and y for a function $f : z = f(x, y)$, where $(x, y) \in \mathbf{R}^2$ with $(x, y, z) \in \mathbf{R}^3$. Given n interpolation points, we want to find an interpolation function Φ with $z = \Phi(x, y)$ defined for $(x, y) \in D$ in such a way that $F(x_i, y_i) = f(x_i, y_i)$ for all $j = 1, \ldots, n$.

If the nodes (x_j, y_j) $(j = 1, \ldots, n)$ do not form a rectangular grid but are arranged in a completely arbitrary and unordered way, we can use spline surfaces or the Shepard method. The Shepard method has proven well suited for the graphic representation of surfaces. Its approximating function Φ is uniquely determined independently from the ordering of the nodes (x_i, y_i) $(i = 1, \ldots, n)$. The function $f : z = f(x, y)$ for $(x, y)D$, where D is an arbitrary region of the Oxy plane, is approximated for the given nodes (x_i, y_i) by the function

$$\Phi(x, y) = \sum_{i=1}^{n} w_i(x, y) \cdot f_i \tag{1}$$

where the weight functions are defined as

$$w_i(x, y) = \frac{(\rho - r_i(x, y))_+^\mu}{\sum_{l=1}^{n} ((\rho - r_l(x, y))_+^\mu} \tag{1a}$$

with the notations

$$r_i(x, y) = \sqrt{(x - x_i)^2 + (y - y_i)^2} \tag{1b}$$

and

$$s_+^\mu = \begin{cases} s^\mu, & s \geq 0 \\ 0, & s < 0 \end{cases} \tag{1c}$$

So we can write

$$\Phi(x, y) = \frac{\sum_{i=1}^{n} ((\rho - r_i(x, y))_+^\mu \cdot f_i}{\sum_{i=1}^{n} ((\rho - r_i(x, y))_+^\mu} \tag{2}$$

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The parameters ρ and μ are determined at the beginning of the interpolation process. The exponent μ can be chosen arbitrarily. If $0 < \mu \leq 1$ the function Φ has peaks at the nodes. If $\mu > 1$ the function is level at the nodes.

The function Φ uses only those nodes (x_j, y_j) within a disc of radius ρ when calculating a new functional value $\Phi(x, y)$, id est, this is a local method. We use a fast local Shepard approximation with Franke-Little weights because of the very reduced complexity order, which is very important for computer graphics applications.

Theorem 1. *Let* $d := \min\{r_j(x_l, y_l) \mid j, l = 1, \ldots, n \text{ and } j \neq l\}$. Φ *(as defined in (2)) is an interpolation function if and only if* $\rho \leq d$.

Proof. The following two conditions are fulfilled for each $(x, y) \in D$:

$$0 \leq w_i(x, y) \leq 1$$
$$\sum_{i=1}^{n} w_i(x, y) = 1$$

The following two conditions are fulfilled if and only if $\rho \leq d$:

$$w_i(x_j, y_j) = 0 \text{ for each } i \neq j \text{ and each } (x, y) \in D;$$
$$w_i(x_i, y_i) = 1 \ (i = 1, \ldots, n). \square$$

2. Improving the Local Shepard Method

We need to analyse three problems when using the Shepard method. The first one is how to organise the input data (the set P) in such a way that we can quickly find all the nodes lying inside of a disc having the centre (x, y) and the radius ρ. This problem can be solved in $\boldsymbol{O}(n \log n)$ pre-processing time, and finding all the necessary nodes needs $\boldsymbol{O}(n \log n + k)$ time, where k is the number of the found nodes ([1], [6]).

The second problem is how to determine an acceptable value for the parameter ρ. We impose for this value the following conditions:

- the disc having the radius ρ must cover at least one node from P, wherever we place this disc in the interior of the $ConvHull(P)$;
- this disc must not cover too much close nodes: we need to quickly compute the value $\Phi(x, y)$.

This problem can be solved in $\boldsymbol{O}(n \log n)$ time, determining the Delaunay diagram of the set P. This diagram is composed by triangles whose vertices are nodes from P, and the interior of the circumcircle of any triangle does not contain any other node from P. We choose the maximum radius of the circumcircle of any triangle: the value of ρ.

The third problem is how to modify the Shepard method in order to obtain an interpolation function. Precisely, we have to answer the following question. Is the below linear equations system (3) uniquely solvable?

$$\Psi(x_i, y_i) := \frac{\sum_{j=1}^{n} (\rho - r_j(x_i, y_i))_+^{\mu} \cdot z_j}{\sum_{j=1}^{n} (\rho - r_j(x_i, y_i))_+^{\mu}} = f_i \ \text{ for } i = 1, \ldots, n \tag{3}$$

Theorem 2. *There exists a positive value* μ_0 *in such a way that for any* $\mu > \mu_0$ *the system (3) is uniquely solvable.*

Proof. Suppose, without loss of generality, that $\rho = 1$ (we can apply a simple scaling operation). By elementary transformations (each equation is multiplied by the denominator of $\Psi(x_i, y_i)$), and because $r_i(x_j, y_j) = r_j(x_i, y_i)$ we obtain that the matrix of (3) has the following properties:

it is symmetric, all the values which lie on the main diagonal are 1, and all the

other values lie in the interval [0,1); but is not necessarily diagonal dominant!

Can we determine a value for μ in such a way so that the system matrix of (3) is diagonal dominant?

Let $s_i(\mu) := \sum_{j=1, j\neq 1}^{n} (1 - r_i(x_j, y_j))_+^{\mu}$. We can determine a value μ so that $s_i(\mu) < 1$, because $s_i(\mu)$ is a continuous, decreasing function (all the terms are less than 1), and $\lim_{\mu\to\infty} s_i(\mu) = 0$. □

To calculate the values $\Psi(x, y)$ we need only those points from P, which lie inside the disc having the centre (x, y) and the radius ρ. How can we find an optimal value (minimum) for the parameter ρ so that

(i) Ψ can be defined at least on $ConvHull(P)$;

(ii) $\Psi(x_i, y_i) = f(x_i, y_i)$ for $i = 1, \ldots, n$; id est, Ψ is indeed an interpolation function.

The proposed algorithm is given below.

Step 1. Determine the Delaunay diagram of the set P. Let considers three non-collinear nodes p_i, p_j, p_k that define a triangle of the diagram, and ρ_{ijk} be the circumcircle of the triangle $p_i p_j p_k$. This circle does not contain any other node - this is an important property of the Delaunay diagram ([1], [3]).

Let $d := \min \text{dist}(p_i, p_j) \mid i, j = 1, \ldots, n$ and $i \neq j$, and $\rho = \max \rho_{ijk}$. The value of d can be quickly determined by scanning all the triangles of the Delaunay diagram. If $\rho \leq d$ then we can easily see that (i) and (ii) are fulfilled.

Step 2. (only if $\rho > d$). In this case only (i) is fulfilled, so we try to solve the system (3). We need to find the values z_i (which are unknown) to fulfil (ii). It is necessary to try different values for the parameter m until the system matrix of (3) is non-singular and well conditioned. At this moment we can solve the system using an iterative method (Jacobi, Gauss-Seidel).

3. Conclusions

The authors of [2] gave a "suggestion" about how to choose the values of the two parameters of the Shepard method:

"To avoid peaks at the nodes, choose $2 \leq \mu \leq 6$. *Our tests indicate that* $0.1 \leq \rho \leq 0.5$ *is the preferred range, where we recommend to choose a small value for* ρ *in case of many available nodes and a larger* ρ *for problems with few nodes.*

For the local method, however, any choice of ρ *near the recommended upper bound of 0.5 leads to unsatisfactory results."*

We don't have any other indication about how to determine these two parameters. The user has to choose manually many couples (ρ, μ) until he gets the expected results.

This is why we gave in this paper an algorithm that automatically determines these two parameters, depending on the topology of the set P. More than that, we improve this method in order to accelerate the computations, and to obtain an interpolation function.

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