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SOLUTIONS TO THE DIOPHANTINE EQUATION $(x + y + z + t)^2 = xyzt$

TITU ANDREESCU

Abstract. The main purpose of this paper is to study the Diophantine equation (2). We will indicate nine different infinite families of positive integral solutions to this equation.

1. Introduction

Generally, integral solutions to equations in three or more variables are given in various parametric forms (see [2] or [6]). In the paper [5] it is proved that the Diophantine equation $x+y+z = xyz$ has solutions in the units of the quadratic field $\mathbb{Q}(\sqrt{d})$ if and only if $d = -1$, 2 or 5 and in these cases all solutions are also given. The problem of finding all integral solutions to this equation remains open. In our paper [1] we constructed different families of infinite integral solutions to the equation

$$(x + y + z)^2 = xyz. \tag{1}$$

We have indicated a general method of generating such families of solutions by using the theory of Pell's equations. The problem of finding all solutions to equation (1) is still open.

In this paper we use the theory of general Pell's equations to generate nine infinite families of positive integral solutions to the equation

$$(x + y + z + t)^2 = xyzt. \tag{2}$$

2. The General Pell's Equation $Ax^2 - By^2 = C$

Recall that the equation

$$u^2 - Dv^2 = 1, \tag{3}$$

where D is a positive integer that is not a perfect square is called Pell's equation.

Denoting by $(u_0, v_0) = (1, 0)$ its trivial solution, the main result concerning equation (3) is the following (see [1], pp. 107-110 or [7]): There are infinitely many solutions in positive integers to (3) and all solutions to equation (3) are given by $(u_n, v_n)_{n\geq 0}$, where

$$\begin{cases} u_{n+1} = u_1u_n + Dv_1v_n \\ v_{n+1} = v_1u_n + u_1v_n \end{cases} \tag{4}$$

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Here (u_1, v_1) represents the fundamental solution to (3), that is the minimal solution different from (u_0, v_0).

It is not difficult to see that (4) is equivalent to

$$u_n + v_n\sqrt{D} = (u_1 + v_1\sqrt{D})^n, \quad n \geq 0. \tag{5}$$

Also, relations (5) could be written in the following useful matrix form:

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} u_1 & Dv_1 \\ v_1 & u_1 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix}, \quad n \geq 0$$

from where

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 & Dv_1 \\ v_1 & u_1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad n \geq 0 \tag{6}$$

From (5) or (6) it follows immediately that

$$\begin{cases} u_n = \dfrac{1}{2}[(u_1 + v_1\sqrt{D})^n + (u_1 - v_1\sqrt{D})^n] \\ \\ v_n = \dfrac{1}{2\sqrt{D}}[(u_1 + v_1\sqrt{D})^n - (u_1 - v_1\sqrt{D})^n], \ n \geq 0 \end{cases} \tag{7}$$

The main method of determining the fundamental solution (u_1, v_1) involves continued fractions. Sometimes this solution is very large, for example if $D = 991$, then

$$\begin{cases} u_1 = 379516400906811930638014896080 \\ v_1 = 12055735790331359447442538767 \end{cases}$$

In what follows we consider the general Pell's equation

$$Ax^2 - By^2 = C, \tag{8}$$

where A, B, C are positive integers with $gcd(A, B) = 1$ and A and B are not perfect squares.

The solvability and unsolvability of equation (8) is discussed in our paper [3]. Concerning this equation we need the following result (see also [4]):

Theorem. *If equation (8) is solvable in positive integers, then it has infinitely many positive integral solutions.*

Proof. We will use the Pell's resolvent associated to equation (8):

$$u^2 - ABv^2 = 1. \tag{9}$$

From the given conditions it follows that AB is not a perfect square so the Pell's equation (9) has infinitely many positive integral solutions. All such solutions are given by (4) or (7), where $D = AB$.

If (x_0, y_0) is a solution to (8) and (u, v) is a solution to (9), then we can construct a new solution to (8) by using the so-called multiplication principle:

$$\begin{cases} x = x_0u + By_0v \\ y = y_0u + Ax_0v \end{cases} \tag{10}$$

Indeed,

$$Ax^2 - By^2 = A(x_0u + By_0v)^2 - B(y_0u + Ax_0v)^2 =$$
$$= (Ax_0^2 - By_0^2)(u^2 - ABv^2) = C \cdot 1 = C.$$

Taking into account that the Pell's resolvent has infinitely many solutions, the conclusion follows. □

In the case when equation (8) is solvable, all of its solutions can be expressed in terms of the solutions to the associated general Pell's equation

$$u^2 - ABv^2 = AC. \tag{11}$$

For more details we refer to [3, Theorem 1] or [8].

3. **Infinite Families of Solutions to Equation (2)**

The transformations

$$x = \frac{u+v}{2} + a, \quad y = \frac{u-v}{2} + a, \quad z = b, \quad t = c \tag{12}$$

where a, b, c are positive integers, bring the equation (2) to the form

$$(u + 2a + b + c)^2 = \frac{bc}{4}(u^2 - v^2) + abcu + a^2bc.$$

Setting the conditions $2(2a+b+c) = abc$ and $bc > 4$, we obtain the following general Pell's equation

$$(bc-4)u^2 - bcv^2 = 4[(2a+b+c)^2 - a^2bc]. \tag{13}$$

There are nine triples (a, b, c) up to permutations satisfying the above conditions: (1,6,4), (1,10,3), (2,2,6), (3,4,2), (3,14,1), (5,2,3), (4,1,9), (7,1,6), (12,1,5).

The following table contains the general Pell's equations (13) corresponding to the above triples (a, b, c), their Pell's resolvent, both equations with their fundamental solutions.

(a, b, c)	General Pell's equation (13) and its fundamental solution	Pell's resolvent and its fundamental solution
(1,6,4)	$5u^2 - 6v^2 = 120,\ (12, 10)$	$r^2 - 30s^2 = 1,\ (11, 2)$
(1,10,3)	$13u^2 - 15v^2 = 390,\ (15, 13)$	$r^2 - 195s^2 = 1,\ (14, 1)$
(2,2,6)	$2u^2 - 3v^2 = 96,\ (12, 8)$	$r^2 - 6s^2 = 1,\ (5, 2)$
(3,4,2)	$u^2 - 2v^2 = 72,\ (12, 6)$	$r^2 - 2s^2 = 1,\ (3, 2)$
(3,14,1)	$5u^2 - 7v^2 = 630,\ (21, 15)$	$r^2 - 35s^2 = 1,\ (6, 1)$
(4,1,9)	$5u^2 - 9v^2 = 720,\ (42, 30)$	$r^2 - 45s^2 = 1,\ (161, 24)$
(5,2,3)	$u^2 - 3v^2 = 150,\ (15, 5)$	$r^2 - 3s^2 = 1,\ (2, 1)$
(7,1,6)	$u^2 - 3v^2 = 294,\ (21, 7)$	$r^2 - 3s^2 = 1,\ (2, 1)$
(12,1,5)	$u^2 - 5v^2 = 720,\ (30, 6)$	$r^2 - 5s^2 = 1,\ (9, 4)$

By using the formula (10) we obtain the following sequences of solutions to equations (13):

$$u_m^{(1)} = 12r_m^{(1)} + 60s_m^{(1)}, \quad v_m^{(1)} = 10r_m^{(1)} + 60s_m^{(1)},$$

where $r_m^{(1)} + s_m^{(1)}\sqrt{30} = (11 + 2\sqrt{30})^m,\ m \geq 1$;

$$u_m^{(2)} = 15r_m^{(2)} + 195s_m^{(2)}, \quad v_m^{(2)} = 13r_m^{(2)} + 195s_m^{(2)},$$

where $r_m^{(2)} + s_m^{(2)}\sqrt{195} = (14 + \sqrt{195})^m,\ m \geq 1$;

$$u_m^{(3)} = 12r_m^{(3)} + 24s_m^{(3)}, \quad v_m^{(3)} = 8r_m^{(3)} + 24s_m^{(3)},$$

where $r_m^{(3)} + s_m^{(3)}\sqrt{6} = (5+2\sqrt{6})^m,\ m \geq 1$;

$$u_m^{(4)} = 12r_m^{(4)} + 12s_m^{(4)}, \quad v_m^{(4)} = 6r_m^{(4)} + 12s_m^{(4)},$$

where $r_m^{(4)} + s_m^{(4)}\sqrt{2} = (3+2\sqrt{2})^m,\ m \geq 1$;

$$u_m^{(5)} = 21r_m^{(5)} + 105s_m^{(5)}, \quad v_m^{(5)} = 15r_m^{(5)} + 105s_m^{(5)},$$

where $r_m^{(5)} + s_m^{(5)}\sqrt{35} = (6+\sqrt{35})^m,\ m \geq 1$;

$$u_m^{(6)} = 42r_m^{(6)} + 270s_m^{(6)}, \quad v_m^{(6)} = 30r_m^{(6)} + 210s_m^{(6)},$$

where $r_m^{(6)} + s_m^{(6)}\sqrt{45} = (161+24\sqrt{45})^m,\ m \geq 1$;

$$u_m^{(7)} = 15r_m^{(7)} + 15s_m^{(7)}, \quad v_m^{(7)} = 5r_m^{(7)} + 15s_m^{(7)},$$

where $r_m^{(7)} + s_m^{(7)}\sqrt{3} = (2+\sqrt{3})^m,\ m \geq 1$;

$$u_m^{(8)} = 21r_m^{(8)} + 21s_m^{(8)}, \quad v_m^{(8)} = 7r_m^{(8)} + 21s_m^{(8)},$$

where $r_m^{(8)} + s_m^{(8)}\sqrt{3} = (2+\sqrt{3})^m,\ m \geq 1$;

$$u_m^{(9)} = 30r_m^{(9)} + 30s_m^{(9)}, \quad v_m^{(9)} = 6r_m^{(9)} + 30s_m^{(9)},$$

where $r_m^{(9)} + s_m^{(9)}\sqrt{5} = (9+4\sqrt{5})^m,\ m \geq 1$.

Formulas (12) yield the following nine families of positive integers solutions to the equation (2):

$$x_m^{(1)} = 11r_m^{(1)} + 60s_m^{(1)} + 1, \quad y_m^{(1)} = r_m^{(1)} + 1, \quad z_m^{(1)} = 6, \quad t_m^{(1)} = 4$$

$$x_m^{(2)} = 14r_m^{(2)} + 195s_m^{(2)} + 1, \quad y_m^{(2)} = r_m^{(2)} + 1, \quad z_m^{(2)} = 10, \quad t_m^{(2)} = 3$$

$$x_m^{(3)} = 10r_m^{(3)} + 24s_m^{(3)} + 2, \quad y_m^{(3)} = 2r_m^{(3)} + 2, \quad z_m^{(3)} = 2, \quad t_m^{(3)} = 6$$

$$x_m^{(4)} = 12r_m^{(4)} + 12s_m^{(4)} + 3, \quad y_m^{(4)} = 3r_m^{(4)} + 3, \quad z_m^{(4)} = 4, \quad t_m^{(4)} = 2$$

$$x_m^{(5)} = 18r_m^{(5)} + 105s_m^{(5)} + 3, \quad y_m^{(5)} = r_m^{(5)} + 3, \quad z_m^{(5)} = 14, \quad t_m^{(5)} = 1$$

$$x_m^{(6)} = 36r_m^{(6)} + 240s_m^{(6)} + 4, \quad y_m^{(6)} = 6r_m^{(6)} + 30s_m^{(6)} + 4, \quad z_m^{(6)} = 1, \quad t_m^{(6)} = 9$$

$$x_m^{(7)} = 10r_m^{(7)} + 15s_m^{(7)} + 5, \quad y_m^{(7)} = 5r_m^{(7)} + 5, \quad z_m^{(7)} = 2, \quad t_m^{(7)} = 3$$

$$x_m^{(8)} = 14r_m^{(8)} + 21s_m^{(8)} + 7, \quad y_m^{(8)} = 7r_m^{(8)} + 7, \quad z_m^{(8)} = 1, \quad t_m^{(8)} = 6$$

$$x_m^{(9)} = 18r_m^{(9)} + 30s_m^{(9)} + 12, \quad y_m^{(9)} = 12r_m^{(9)} + 12, \quad z_m^{(9)} = 1, \quad t_m^{(9)} = 5.$$

Remarks. 1) In [9] only solution $(x_m^{(7)}, y_m^{(7)}, z_m^{(7)}, t_m^{(7)})$ is found.

2) Note the atypical form of solution $(x_m^{(6)}, y_m^{(6)}, z_m^{(6)}, t_m^{(6)})$.

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American Mathematics Competitions, University of Nebraska – Lincoln, 1740 Vine Street, Lincoln, NE 68508, USA
E-mail address: titu@amc.unl.edu