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# ASYMPTOTICAL VARIANTS OF SOME FIXED POINT THEOREMS IN ORDERED SETS

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**Abstract**. In this paper we will present some fixed point theorems in ordered sets with condition for operator and its iterates too.

## 1. Introduction

Let $(X, \leq)$ be an ordered set $(X \neq \emptyset)$ and $f : X \to X$ an operator. We denote by

$$F_f = \{x \in X : f(x) = x\}$$

the fixed point set of $f$.

In this note we need the following results [1-7].

**Theorem of Tarski.** *Let $(X, \leq)$ be a complete lattice, $f : X \to X$ an increasing operator. Then $F_f \neq \emptyset$ and $(F_f, \leq)$ is a complete lattice.*

**Theorem of Birkhoff-Bourbaki.** *Let $(X, \leq)$ be right inductive ordered set and let $f : X \to X$ be an expansive operator. Then $F_f \neq \emptyset$.*

**Lemma.** *Let $X$ be nonempty set and $f, g : X \to X$ two commuting operators. Then:*

(i) *$F_g = \emptyset$ or $F_g \in I(f)$;*
(ii) *$F_f = \emptyset$ or $F_f \in I(g)$;*

## 2. The main results

**Theorem 1.** *Let $(X, \leq)$ be a an ordered set and $f : X \to X$ an increasing operator. We suppose that there exist $k \in \mathbb{N}^*$ and $Y \subset X$ such that:*

(a) $f^k(X) \subset Y$;
(b) *$(Y, \leq)$ is a complete lattice.*

*Then $F_f \neq \emptyset$.*

**Proof.** From (a) and (b) we have that the restriction of iterate $f^k$ has the following properties $f^k|_Y : Y \to Y$ and $f^k$ is an increasing operator.

$f$ is an increasing operator, i.e. for any $x, y \in X$ we have

$$x \leq y \Longrightarrow f(x) \leq f(y)$$

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$$f(x) \leq f(y) \Longrightarrow f(f(x)) \leq f(f(y))$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$f^{k-1}(x) \leq f^{k-1}(y) \Longrightarrow f^k(x) \leq f^k(y).$$

Since $(Y, \leq)$ is a complete lattice, from Theorem of Tarski it follows that $F_{f^k} \neq \emptyset$ and $\left(F_{f^k}, \leq\right)$ is a complete lattice. Because $f$ and $f^k$ are commuting operators, from Lemma we have that $f\left(F_{f^k}\right) \subset F_{f^k}$.

We apply the Tarski Theorem to $f : F_{f^k} \to F_{f^k}$ and we conclude that there exists at least a fixed point $\left(\in F_{f^k}\right)$ which means that $F_f \neq \emptyset$.□

**Theorem 2.** *Let $(X, \leq)$ be a an ordered set and $f : X \to X$ be an expansive operator. We suppose that there exist $k \in \mathbb{N}^*$ and $Y \subset X$ such that:*

(a) $f^k(X) \subset Y$;

(b) $(Y, \leq)$ *is a right inductive ordered set.*

*Then $F_f \neq \emptyset$.*

**Proof.** From (a) we have $f^k|_Y : Y \to Y$. Since $f$ is an expansive operator, i.e.

$$x \leq f(x), \quad \forall x \in X,$$

we obtain

$$x \leq f(x) \leq f(f(x)) = f^2(x) \leq \dots \leq f^{k-1}(x) \leq f^k(x),$$

which means that $f^k$ is an expansive operator. From Theorem of Birkhoff-Bourbaki we have that $F_{f^k} \neq \emptyset$. Let $x^* \in F_{f^k}$, we want to prove that $x^* \in F_f$.

Suppose that $x^*$ is not a fixed point of $f$: $f(x^*) \neq x^*$. We have two cases: $x^* < f(x^*)$ and $x^* > f(x^*)$.

**Case I:** $x^* < f(x^*)$

Since $f$ is an expansive operator we deduce

$$x^* < f(x^*) \leq f^2(x^*) \leq \dots \leq f^{k-1}(x^*) \leq f^k(x^*) = x^*,$$

which is a contradiction.

**Case II:** $x^* > f(x^*)$

$$x^* > f(x^*) \geq f^2(x^*) \geq \dots \geq f^{k-1}(x^*) \geq f^k(x^*) = x^*,$$

which is also a contradiction.

Thus we have that $x^* \in F_f$.□

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