STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVIII**, Number 1, March 2003

ON CERTAIN CLASSES OF GENERALIZED CONVEX FUNCTIONS WITH APPLICATIONS, II

J. SÁNDOR

Dedicated to Professor Wolfgang W. Breckner at his 60^{th} *anniversary*

In the first part [8] we have studied the η-invex functions first introduced by the author in 1988. We have also introduced and studied η-invexity, η-pseudo-invexity, Jensen-invexity (and the underlying invex and Jensen-invex sets), almost-invexity, as well as almost-cvazi-invexity.

In this second part we shall introduce and study the notions of A-convexity; resp. Λ-invexity ($\Lambda \subset [0,1]$, dense).

1. *A*-convex functions

Definition 1.1. ([5]) Let X be a real linear space, and $B : X \times X \to \mathbb{R}$ a given application. We say that a function $f : X \to \mathbb{R}$ is B**-subadditive** (superadditive) if one has

$$f(x+y) \leq (\geq) f(x) + f(y) + B(x,y) \text{ for all } x, y \in X. \tag{1}$$

An immediate property related to this definition is:

Proposition 1.1. *If B is an antisymmetric application and f is B-subadditive (superadditive), then f is subadditive (superadditive).*

Proof. One can write

$$f(x+y) \leq f(x) + f(y) + B(x,y) \text{ and } f(x+y) \leq f(y) + f(x) + B(y,x)$$

By addition, it follows

$$f(x+y) \leq f(x) + f(y) + \frac{1}{2}[B(x,y) + B(y,x)] = f(x) + f(y),$$

since $B(x,y) = -B(y,x)$, B being antisymmetric. Therefore, f is subadditive.

Received by the editors: 30.09.2002.

Definition 1.2. Let $B : X \times X \to \mathbb{R}_+$, with X again a real linear space. We say that $f : X \to \mathbb{R}$ is **absolutely-B-subadditive**, if the following relation holds true:

$$|f(x+y) - f(x) - f(y)| \le B(x,y) \tag{2}$$

Theorem 1.1. [5] *If $B : X \times X \to \mathbb{R}$ is homogeneous of order zero, and if $f : X \to \mathbb{R}$ is absolutely-B-subadditive, then there exists a single additive function $g : X \to \mathbb{R}$, which "quadratically approximates" f, i.e.*

$$|f(x) - g(x)| \le B(x,x), \quad x \in X \tag{3}$$

Proof. Put $x := 2^{n-1}x$, $y := 2^{n-1}x$ in relation (2). We get

$$\left| \frac{f(2^n x)}{2^n} - \frac{f(2^{n-1}x)}{2^{n-1}} \right| \le \frac{B(x,x)}{2^n}.$$

By the modulus inequality, one has, on the other hand

$$\left| \frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m} \right| \le \left| \frac{f(2^n x)}{2^n} - \frac{f(2^{n-1}x)}{2^{n-1}} \right| + \left| \frac{f(2^{n-1} x)}{2^{n-1}} - \frac{f(2^{n-2}x)}{2^{n-2}} \right| +$$

$$+ \cdots + \left| \frac{f(2^{m+1} x)}{2^{m+1}} - \frac{f(2^m x)}{2^m} \right| \text{ for } n > m.$$

Thus

$$\left| \frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m} \right| \le B(x,x) \left(\frac{1}{2^n} + \frac{1}{2^{n-1}} + \cdots + \frac{1}{2^m} \right)$$

This inequality easily implies that the sequence of general term $x_n = \dfrac{f(2^n x)}{2^n}$ is fundamental. $\mathbb{R}$ being a complete metric space, (x_n) has a limit; let

$$g(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{4}$$

We now prove that g is additive. Indeed, one has

$$|g(x+y) - g(x) - g(y)| = \lim_{n\to\infty} \left| \frac{f(2^n x + 2^n y)}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n} \right| \le$$

$$\le \lim_{n\to\infty} \frac{B(x,y)}{2^n} = 0.$$

This gives $g(x+y) = g(x) + g(y)$. We now show that g is unique. Let us assume that there exists another additive application h such that

$$|f(x) - h(x)| \le B(x,x).$$

Then

$$|g(x) - h(x)| = |g(x) - f(x) + f(x) - h(x)| \leq 2B(x, x),$$

by assumption. Thus

$$|g(2^n x) - h(2^n x)| \leq 2B(2^n x, 2^n x),$$

implying

$$|g(x) - h(x)| \leq \frac{B(x, x)}{2^{n-1}} \to 0$$

as $n \to \infty$. (Indeed, $g(2^n x) = 2^n g(x)$ and $h(2^n x) = 2^n h(x)$; g and h being additive).

Now, an inductive argument shows that $|f(2^n x) - 2^n f(x)| \leq 2^n B(x, x)$. By dividing with 2^n and letting $n \to \infty$, one has $|f(x) - g(x)| \leq B(x, x)$, i.e. g approximates f in the above defined manner.

Proposition 1.2. *Let* $f : (0, +\infty) \to \mathbb{R}$ *be such that the application* $x \to \dfrac{f(x)}{x}$ *is* B*-decreasing on* $(0, +\infty)$*. Then* f *is* B_1*-subadditive, where*

$$B_1(x, y) = xB(x + y, x) + yB(x + y, y); \quad x, y \in (0, +\infty).$$

Proof. Since $x, y > 0$; $x + y > x$ implies

$$\frac{f(x + y)}{x + y} \leq \frac{f(x)}{x} + B(x + y, x)$$

and

$$\frac{f(x + y)}{x + y} \leq \frac{f(y)}{y} + B(x + y, x)$$

(here $x + y > y$). Therefore,

$$f(x + y) = \frac{f(x + y)}{x + y}(x + y) \leq \frac{f(x)}{x} \cdot x + xB(x + y, x) + \frac{f(y)}{y} \cdot y + yB(x + y, y) =$$

$$= f(x) + f(y) + B_1(x, y),$$

by the above written two inequalities, and by the definition of B_1.

Definition 1.3. Let Y be a **convex subset** of the real linear space X. Let $A : Y \times Y \times Y \to \mathbb{R}$ be an application of three variables. We say that the function $f : Y \to \mathbb{R}$ is A**-convex** (concave) if the following inequality holds true:

$$f(\lambda u + (1 - \lambda)v) \leq (\geq) \lambda f(u) + (1 - \lambda) f(v) +$$

$$+\lambda(u - v)A(\lambda u + (1 - \lambda)v, u, v) \tag{5}$$

for all $u, v \in Y$, all $\lambda \in [0,1]$.

Definition 1.4. Let Y be an η-invex set of X (see [8] for definition and related examples or results). We say that $f : Y \to \mathbb{R}$ is an $\eta - A$**-invex** (incave) function, if

$$f(v + \lambda\eta(u,v)) \leq (\geq) \lambda f(u) + (1-\lambda)f(v) + \lambda(u-v)A(\eta(u,v),u,v) \tag{6}$$

for all $u, v \in Y$, all $\lambda \in [0,1]$.

Proposition 1.3. *Let $A : \mathbb{R}^3_+ \to \mathbb{R}$ and $f : \mathbb{R}_+ \to \mathbb{R}$ be an $A(\cdot,\cdot,0)$-concave function. Put $A_1(\cdot,\cdot) = A(\cdot,\cdot,0)$ and assume that $f(0) = 0$. Then f is a B_1-subadditive function, where*

$$B_1(x,y) = -xA_1(x,x+y) - yA_1(y,x+y). \tag{7}$$

Proof. First remark that the A-convexity (concavity) of f is equivalent to the inequality

$$\frac{f(x)-f(z)}{x-z} \leq (\geq) \frac{f(y)-f(z)}{y-z} + A(x,y,z), \quad x < z < y \tag{8}$$

where the application $F_z(x) = \dfrac{f(x)-f(z)}{x-z}$ is an A_z-increasing application for all fixed z, with $A_z(x,y) = A(x,y,z)$. Indeed, let $z < x < y$. Then inequality (8) with $\geq$ can be written also as

$$(y-z)f(x) - (y-z)f(z) \geq (x-z)f(y) - (x-z)f(z) + (x-z)(y-z)A(x,y,z),$$

i.e.

$$(y-z)f(x) \geq (x-z)f(y) + (y-x)f(z) + (x-z)(y-z)A(x,y,z)$$

or

$$f(x) \geq \lambda f(y) + (1-\lambda)f(z) + (x-z)A(x,y,z),$$

with $\lambda := \dfrac{x-z}{y-z} \in (0,1)$ and $1-\lambda = 1 - \dfrac{x-z}{y-z} = \dfrac{y-x}{y-z}$ and $x = \lambda y + (1-\lambda)z$. Since, by assumption one has $f(0) = 0$ and $\dfrac{f(x)-f(0)}{x-0} = \dfrac{f(x)}{x}$, from the above remark, the function $\dfrac{f(\cdot)}{(\cdot)}$ is A_1-increasing. Thus, one can write

$$\frac{f(x)}{x} \geq \frac{f(x+y)}{x+y} + A_1(x,x+y), \text{ resp.}$$

$$\frac{f(y)}{y} \geq \frac{f(x+y)}{x+y} + A_1(y, x+y),$$

giving

$$f(x) + f(y) \geq f(x+y)\left(\frac{x}{x+y} + \frac{y}{x+y}\right) + xA_1(x, x+y) + yA_1(y, x+y) =$$

$$= f(x+y) - B_1(x, y).$$

This implies $f(x+y) \leq f(x) + f(y) + B_1(x, y)$, i.e. f is B_1-subadditive, where B_1 is given by (7).

Proposition 1.4. *Let $f : (0, \infty) \to \mathbb{R}$ be a convex function (in the classical sense) and B-subadditive. Then the function g given by $g(x) = \dfrac{f(x)}{x}$ is a C-increasing function for some $C : (0, \infty) \times (0, \infty) \to \mathbb{R}$.*

Proof. Let $\lambda = \dfrac{x}{x+h} \in (0, 1)$ with $h > 0$ and $x + h = \lambda x + (1 - \lambda)(2x + h)$. From the B-subadditivity of f one has

$$f(2x + h) \leq f(x) + f(x + h) + B(x, x + h).$$

The convexity of f implies

$$f(x + h) \leq \lambda f(x) + (1 - \lambda)f(2x + h).$$

Therefore,

$$f(x + h) \leq \lambda f(x) + (1 - \lambda)f(x) + (1 - \lambda)f(x + h) + (1 - \lambda)B(x, x + h).$$

This gives

$$\lambda f(x + h) \leq f(x) + (1 - \lambda)B(x, x + h).$$

Here $\lambda = \dfrac{x}{x+h}$ and $1 - \lambda = \dfrac{h}{x+h}$, so

$$\frac{x}{x+h} f(x + h) \leq f(x) + \frac{h}{x+h} B(x, x + h),$$

or

$$\frac{f(x+h)}{x+h} \leq \frac{f(x)}{x} + C(x, h),$$

where $C(x, h) = \dfrac{h}{x} \cdot \dfrac{B(x, x+h)}{x+h}$, which concludes of the proof of the C-monotonicity of g.

2. Λ-invex functions ($\Lambda \subseteq [0, 1]$, dense)

Let $\Lambda \subseteq [0, 1]$ be a fixed, dense subset of $[0, 1]$. As a generalization of the notion of η-cvazi-invexity (see [8]), we shall introduce the notion of $\eta - \Lambda$**-invexity** as follows:

Definition 2.1. ([7]) Let X be a real linear space, $S \subset X$ an η-invex subset of X, where $\eta : X \times X \to X$ (see [8]), and let $f : S \to \mathbb{R}_\infty = \mathbb{R} \cup \{+\infty\}$. We say that f is an $\eta - \Lambda$**-invex** function, if the following inequality holds true:

$$f(x + \lambda\eta(y, x)) \leq \max\{f(x), f(y)\} \text{ for all } x, y \in S, \text{ all } \lambda \in \Lambda. \quad (9)$$

Remark 2.1. When $\Lambda \equiv [0, 1]$, the notion of $\eta - \Lambda$-invexity of f coincides with that of η-cvazi-invexity of f.

Definition 2.2. The set $D(f) = \{x \in S : \ f(x) < +\infty\}$ will be called the **effective domain** of $f : S \to \mathbb{R}_+$.

Definition 2.3. A point $x \in S$ with the property $f(x) = +\infty$ will be called as a **singular point** of f. The **set of all singular points** of f will be denoted by $S(f)$.

In what follows we shall assume that $S = X$, which is a **real normed space**. Let us use the following (standard) notations

$$\underline{f}(x) = \liminf_{y \to x} f(y); \quad \overline{f}(x) = \limsup_{y \to x} f(y).$$

The following result extends theorems due to F. Bernstein and G. Doetsch [1], E. Mohr [4], A. Császár [2].

Theorem 2.1. ([7]) *Let $f : X \to \mathbb{R}_\infty$ be an $\eta - \Lambda$-invex set and let $K \subset D(f)$ be an open, η-invex set. Let us assume that the application $\eta : X \times X \to X$ is continuous in the strong topology and that $\underline{f}(x) > -\infty$ for all $x \in X$. Then the function $\underline{f} : K \to \mathbb{R}$ is η-cvazi-invex.*

Proof. Let $x, y \in K$. There exists $b \in (0, 1)$ with $z = x + b\eta(y, x) \in K$. Since we are in the case of normed spaces, we can select sequences (x_k), (y_k) such that $x_k \to x$, $y_k \to y$ $(k \to \infty)$ imply $f(x_k) \to \underline{f}(x)$ and $f(y_k) \to \underline{f}(y)$ $(k \to \infty)$.

Let then $(a_k) \subset \Lambda$ be a sequence such that $a_k \to b$, and put $z_k = x_k + a_k\eta(y_k, x_k)$.

The function η being continuous in the norm topology, one can write $z_k \to x + b\eta(y,x) = z$ and $\underline{f}(x) \leq \liminf_{k\to\infty} f(z_k)$. But from $f(z_k) \leq \max\{f(x_k), f(y_k)\}$, by taking $k \to \infty$ one obtains immediately

$$\underline{f}(z) \leq \liminf_{k\to\infty} f(z_k) \leq \max\left\{\liminf_{k\to\infty} f(x_k), \liminf_{k\to\infty} f(y_k)\right\} =$$
$$= \max\{\underline{f}(x), \underline{f}(y)\},$$

proving the η-cvazi-invexity of the function $\underline{f}$.

Proposition 2.1. *If $f : X \to \mathbb{R}_\infty$ is η-invex (or η-cvazi-invex), then the set $D(f)$ is η-invex set (or η-cvazi-invex set).*

Proof. Let $x, y \in D(f)$. Then $f(x) < +\infty$, $f(y) < +\infty$, so

$$f(x + \lambda\eta(y,x)) \leq \lambda f(y) + (1-\lambda)f(y) < +\infty$$

(in the η-invex case); or

$$f(x + \lambda\eta(y,x)) \leq \max\{f(x), f(y)\} < +\infty$$

(in the η-cvazi-invex case). In any case, one has $x+\lambda\eta(y,x) \in D(f)$ for all $x, y \in D(f)$, all $\lambda \in [0,1]$, proving the η-invexity of the set $D(f)$.

Theorem 2.2. *Let us assume that the real Banach space X and the application η have the following property:*

For $M \subset X$, if $x, x_0 \in intM_0$, then there exists $\lambda \in (0,1)$ and $y \in M$ such that

$$x = x_0 + \lambda\eta(y, x_0). \tag{$*$}$$

Let $f : X \to \mathbb{R}_\infty$ be an $\eta - \Lambda$-invex function and let $x_0 \in intD(f)$ be selected such that $\overline{f}(x_0) < +\infty$. If η is nonexpansive related to the second argument; then $\overline{f}(x) < +\infty$ for all $x \in intD(f)$.

Proof. Let $M := D(f)$ in $(*)$ and let $x, x_0 \in D(f)$, where $\overline{f}(x) = +\infty$, $\overline{f}(x_0) < +\infty$. By condition $(*)$, there exists $\lambda \in \Lambda$ and $y \in D(f)$ such that

$$x = x_0 + \lambda\eta(y, x_0). \tag{10}$$

Select now a sequence (x_k) with $x_k \in D(f) \setminus \{x\}$ such that $x_k \to x$, $f(x_k) \to +\infty$ $(k \to +\infty)$. Thus there exists $k_0 \in \mathbb{N}$ with

$$f(x_k) > f(y) \text{ for all } k \geq k_0. \tag{11}$$

Let z_k be determined by the equation

$$x_k = z_k + \lambda\eta(y, z_k), \quad k \in \mathbb{N}. \tag{12}$$

Equation (10) can be solved for all z_k (k=fixed), since, by letting, with $z_k = z$, the application $g(z) = x - \lambda\eta(y, z)$, $g : X \to X$ becomes a **contraction**. Indeed, one has

$$\|g(z_1) - g(z_2)\| = \lambda\|\eta(y, z_1) - \eta(y, z_2)\| \leq \lambda < 1,$$

η being nonexpansive upon the second argument.

Now Banach's classical contraction principle assures the existence of a unique fix point of the operator g; in other words, equation (10) has a single solution.

We shall prove now that

$$z_k \to x_0. \tag{13}$$

For this aim, remark that

$$\|x_k - x\| = \|z_k - x + \lambda\eta(y, z_k)\| =$$

$$= \|z_k - x_0 + \lambda(\eta(y, x_0) - \eta(y, z_k))\| > \|z_k - x_0\| - \lambda\|\eta(y, x_0) - \eta(y, z_k)\| >$$

$$> \|z_k - x_0\| - \lambda\|z_k - x_0\| = (1 - \lambda)\|z_k - x_0\|.$$

Therefore,

$$\|z_k - x_0\| < \frac{1}{1 - \lambda}\|x_k - x\| \to 0$$

as $k \to \infty$, finishing the proof of relation (14).

Let now z_k be defined uniquely by (10), and let $k \geq k_0$ be given by (11). One can write

$$f(y) < f(x_k) \leq \max\{f(z_k), f(y)\} = f(z_k),$$

so on base of (13), one obtains $\overline{f}(x_0) \geq \lim_{k\to\infty} f(z_k) = +\infty$, which contradicts the assumption $\overline{f}(x_0) = +\infty$.

Remark 2.2. If η has the **nonexpansivity property upon both arguments**, i.e.

$$\|\eta(y, x) - \eta(y_0, x_0)\| \leq \|y - y_0\| + \|x - x_0\|,$$

it is immediately seen that if $M \subseteq X$ is an invex set, then $intM$ will be also invex (for the same η; i.e. η-invex). Thus, for $\Lambda \equiv [0, 1]$, on base of Proposition 2.1, relation $(*)$

holds true for η-cvazi-invex sets. Remark that for $y = y_0$, the nonexpansivity upon the second variable is contained in the above duble nonexpansivity property.

We now prove the main result of this section:

Theorem 2.3. ([6], [7]) *Let us assume that $f : X \to \mathbb{R}_\infty$ satisfies the conditions of Theorem 2.2 and that f is* **inferior semicontinuous**. *In this case one has the following alternatives: i) $D(f) = \emptyset$, ii) If there exists $x_0 \in intD(f)$ with $\overline{f}(x_0) < +\infty$; then the set $S(f)$ of singularities can be written as a numerable intersection of dense sets in X. If $intD(f) \neq \emptyset$, then $\overline{f}(x) < +\infty$ for all $x \in intD(f)$.*

Proof. For $n \in \mathbb{N}$ defined the sets $X_n = \{x \in X : \ f(x) > n\}$, which is an open set. One can write: $S(f) = \cap\{X_n : \ n \in \mathbb{N}\}$. The sets X_n are dense in X, since if not, i.e. if X_{n_0} is not dense ($n_0 \in \mathbb{N}$), then there exists $y_0 \in X$ and a closed ball $B(y_0, r) = B$ such that $B \cap X_{n_0} = \emptyset$. Thus for $x \in B$ we would have $f(x) \leq n_0$. If $intD(f) \neq \emptyset$, by Theorem 2.2 we have $\overline{f}(x) < +\infty$ for all $x \in intD(f)$, which is impossible, by assumption. If $\overline{f}(x_0) = +\infty$ for an $x_0 \in intD(f)$, by Baire's classical lemma one has $S(f) = \cap\{X_n : \ n \in \mathbb{N}\}$ is dense in X. There for $intD(f) = \emptyset$, contradicting $x_0 \in intD(f)$.

Remark 2.3. Theorem 2.3 constitutes a generalization of a theorem by J. Kolumbán [3]. For $\eta(x, y) = x - y$ (i.e. the convex case), we can deduce a generalization of the well known theorem of Banach-Steinhaus on the condensation of singularities.

References

[1] F. Bernstein and G. Doetsch, *Zur Theorie der konvexen Funktionen*, Math. Ann. **76**(1915), 514-516.
[2] A. Császár, *On convex sets and functions* (Hungarian), Mat. Lapok (Budapest) **9**(1958), 273-282.
[3] J. Kolumbán, *Verallgemeinerte konvexe Funktionen und das Prinzip der locallen beschränktheit*, L'Analyse Numér. Th. Approx. (Cluj), **11**(1982), 99-108.
[4] E. Mohr, *Beitrag zur Theorie der konvexen Funktionen*, Math. Nachr. **8**(1952), 133-148.
[5] J. Sándor, *Some open problems in the theory of functional equations*, Simpozium on applications of functional equations in education, science and industry, 4^{th} june, 1988, Odorheiu-Secuiesc, Romania.
[6] J. Sándor, *On the principle of condensation of singularities*, Fourth National Conf. on Mathematical Inequalities, Sibiu, 30-31 oct. 1992.
[7] J. Sándor, *Generalized invexity and its applications in optimization theory*, First joint Conf. on Modern applied Analysis, Ilieni (Romania), 12-17 June 1995.
[8] J. Sándor, *On certain classes of generalized convex functions with applications*, I, Studia Univ. Babeş-Bolyai, Math., **44**(1999), 73-84.

Faculty of Mathematics and Computer Science,
Babeş-Bolyai University, 3400 Cluj-Napoca, Romania