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CONSTRAINT CONTROLLABILITY IN INFINITE DIMENSIONAL BANACH SPACES

MARIAN MUREŞAN

Dedicated to Professor Wolfgang W. Breckner at his 60^{th} *anniversary*

Abstract. Some well known criteria of controllability of linear and time invariant systems in $\mathbb{R}^n$ has been extended in various directions. First we review briefly this topic. Then we introduce a necessary and sufficient criterion of approximately locally null-controllability for a system of differential equations in infinite dimensional Banach spaces. Several comments end the paper.

Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. Denote by W an open neighborhood of a point $x_0 \in \mathbb{R}^n$. Consider the following system of differential equations

$$x'(t) = f(t, x(t), u(t)), \qquad x(t_0) = x_0,\ t \in T \tag{1}$$

where T is an interval (bounded or not), $t_0 \in T$, $T \ni t \mapsto x(t) \in \mathbb{R}^n$ is the state trajectory, and $T \ni t \mapsto u(t) \in U \subset \mathbb{R}^m$ is the control function.

Example. If f is a linear functions and the dynamics of system (1) is time invariant, we get the simplest case

$$x'(t) = Ax(t) + Bu(t), \qquad A \in M_{n\times n},\ B \in M_{n\times m}. \tag{2}$$

Roughly speaking, (1) is said to be *controllable* if every state is accessible from every other state.

We mention some topics and works related to the idea of controllability

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- controllability in the time invariant case in finite dimensional spaces, [?], [?] and the references therein;
- controllability in the non-linear case in finite or infinite dimensional spaces, fixed point method, [?], [?], [?], [?], [?];
- controllability of convex processes in finite dimensional spaces, [?], [?], [?], [?];
- constraint controllability in Banach spaces, [?], [?], [?], [?], [?], [?], [?], [?], [?], [?];
- approximate null controllability of certain differential inclusions in infinite dimensional Banach spaces, [?].

1. **Linear case in finite dimensional spaces**

In this case we have system (2), i.e.,

$$x'(t) = Ax(t) + Bu(t), \qquad A \in M_{n\times n},\ B \in M_{n\times m}.$$

If the control function u is (at least) Lebesgue integrable, the general solution of the above system is

$$x(t) = e^{At}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau, \quad t \in T. \tag{3}$$

Following [?] we say that system (1) is *(completely) state*

(i) *approximately controllable* on the finite interval $[t_0, t_f] \subset T$ if given $\varepsilon > 0$ and two arbitrary initial and final points x_0 and x_f in the state space there is an admissible controller $u(\cdot)$ on $[t_0, t_f]$ steering x_0, along a solution curve of (1), to an ε-ball of x_1, that is such that $\|x(t_f, t_0, x_0, u) - x_1\| \leq \varepsilon$.

(ii) *exactly controllable* on $[t_0, t_f]$ if $\varepsilon = 0$ in (i).

To system (2) we introduce the so-called *controllability Gramian*

$$G(t_0, t_f) = \int_{t_0}^{t_f} e^{A(t_f-\tau)}BB^T e^{A^T(t_f-\tau)}d\tau, \tag{4}$$

and the *controllability matrix*

$$Q = [B, AB, A^2B, \cdots, A^{n-1}B]. \tag{5}$$

It is well-known the next characterization theorem

Theorem 1.1. *For the linear time invariant system* (2) *the following statements are equivalent*

(a) (2) *is completely controllable;*

(b) *the controllability Gramian satisfies* $G(t_0, t) > 0$ *for all* $t > t_0$;

(c) *the controllability matrix* Q *has rank* n *(Kalman criterion);*

(d) *the rows of* $e^{At}B$ *are linearly independent functions of time;*

(e) *the rows of* $(sI - A)^{-1}B$ *are linearly independent functions of* s;

(f) $rank([A - \lambda I, B]) = n$, *for all* λ *(suffices to check only the eigenvalues of* A*);*

(g) $v^T B = 0$ *and* $v^T A = \lambda v^T \implies v = 0$ *(Popov-Belevich-Hautus test);*

(h) *given any set* Γ *of numbers in* $\mathbb{C}$ *there exists a matrix* K *such that the spectrum of* $A + BK$ *is equal to* Γ *(pole placement condition).*

2. **The result**

In order to present our result we introduce some notations. Let Z be a topological space and $Y \subset Z$. By $\operatorname{int} Y$ and $\operatorname{cl} Y$ we denote the set of interior points, and the closure of Y, respectively. Let Z be a linear space and $Y \subset Z$, then by $\operatorname{co} Y$ we denote the convex hull of Y. If X is a Banach space, then by $\mathcal{L}(X)$ we denote the space of linear and bounded operators from X in X. X^* is the Banach space of the linear and continuous functionals on X. Let F be a multifunction from a σ-algebra to a topological space. By S_F we denote the set of measurable selections from F. Under convenient assumptions, by S^1_F we denote the set of Bochner integrable selections from F, see [**?**], [**?**], [**?**].

Consider a real interval $T := [t_0, t_f]$ with $t_0 < t_f$ and μ the Lebesgue measure on T. Let X and Y be separable real Banach spaces. Let $\mathrm{B}_\delta = \{x \in X \mid \|x\| \leq \delta\}$. We denote the closed unit ball by B, too. We consider further

(U) a weakly measurable multifunction $U : T \rightsquigarrow Y$ having nonempty and closed values;

(B) a Carathéodory mapping $B : T \times Y \to X$ (measurable in the first variable and continuous in the second one) such that there exits a positive integrable function

m defined on T satisfying

$$U(t,u) \subset m(t)\mathrm{B}, \quad \text{for all} \quad t \in T, \quad u \in U(t). \tag{6}$$

(A) a family $\{A(t)\}_{t\in T}$ of linear and densely defined operators generating an evolution operator $S : \Delta = \{(t,s) \in T \times T \mid t_0 \le s \le t \le t_f\} \to \mathcal{L}(X)$, i.e.

$S(t,t) = I$, $\forall t \in T$, I is the identity,

$S(t,\tau)S(\tau,s) = S(t,s)$, $\forall\ t_0 \le s \le \tau \le\le t \le t_f$,

$S : \Delta \to \mathcal{L}(X)$ is continuous in the strong operator topology, [**?**].

Also, $B(t,U(t)) := \{x \in X \mid \exists\ u \in U(t)$ with $x = B(t,u)\}$. For $M \subset X$, $M \ne \emptyset$, the support function $\sigma_M(\cdot)$ of M is defined by

$$\sigma_M(x^*) = \sup_{x\in M}(x^*,x) = \sup_{x\in M} x^*(x) = \sigma(x^*(M)), \quad x^* \in X^*.$$

Under the above conditions our attention focuses on the following system

$$x'(t) = A(t)x(t) + B(t,u(t)), \quad t \in T,\ u \in S_U. \tag{7}$$

Throughout the present paper we are interested in some properties of the mild solutions of the system (7), i.e. given $x_0 \in X$ (as initial value) a mild solution of (7) is a continuous function $x \in C(T,X)$ which can be written as

$$x(t) = S(t,t_0)x_{t_0} + \int_{t_0}^{t} S(t,s)B(s,u(s))ds,\ t \in T, \tag{8}$$

where u is a measurable selection of the multifunction U such that $B(\cdot,u(\cdot)) \in L^1$.

The reachable set from x_0 at time $t \in T$ is defined as

$$R(t,x_0) = \{x(t) \in X \mid x(\cdot) \text{ is a mild solution of (7)}\}.$$

Different notions of controllability are investigated in [**?**] and [**?**]. We now recall here only one in [**?**]. System (7) is said to be *approximately locally null-controllable* if there exists an open neighborhood V of the origin such that for all $x_0 \in V$, $0 \in \mathrm{cl}(R(t_f,x_0))$.

Remarks 2.1.

(a) From (U) it follows that $S_U \ne \emptyset$; moreover, from the Castaing representation theorem, [**?**, theorem 5.6], [**?**, theorem 4.2.3], or [**?**, p. 76] it follows that there exists a countable family of measurable functions $\{u_n\}_{n\ge 1}$ such that $U(t) = \mathrm{cl}\{u_n(t) \mid n \ge 1\}$, for all $t \in T$.

(b) The multifunction U has closed values. Then, by [**?**, theorem 6.5] the multifunction $T \ni t \mapsto B(t, U(t))$ is weakly measurable. Since $B(t, U(t)) \subset m(t)\mathrm{B}$, $t \in T$, and each mapping $B(\cdot, u_n(\cdot))$ is a measurable selection of $B(\cdot, U(\cdot))$, we conclude that the multifunction $B(\cdot, U(\cdot))$ has a family $(B(\cdot, u_n(\cdot)))_n$ of integrable selections. Thus the definition of mild solution in (8) makes sense and the reachable set is nonempty.

(c) The mapping $T \times Y \ni (t, u) \mapsto S(t_f, t)B(t, u) \in X$ is Carathéodory. As above we conclude that the multifunction

$$[t_0, t] \ni s \mapsto S(t, s)B(s, U(s))$$

is weakly measurable, for all $t \in [t_0, t_f]$.

Theorem 2.1. *Suppose the assumptions* (U), (B), and (A) *are satisfied. Then*

(a) *if* $S(t_f, t)B(t, U(t)) \neq \{0\}$ *on a set of positive Lebesgue measure and* (7) *is approximately locally null-controllable, then there exists* $x^* \in X^* \setminus \{0\}$ *and* $E \subset T$ *Lebesgue measurable such that*

$$\mu(E) > 0, \text{ and } 0 < \sigma(x^*(S(t_f, t)B(t, U(t)))), \quad \forall\, t \in E;$$

(b) *if* $0 \in B(t, U(t))$ *a.e. and for every* $x^* \in X^* \setminus \{0\}$ *there exists* $E \subset T$ *Lebesgue measurable with* $\mu(E) > 0$ *such that for all* $t \in E$ $\sigma(x^*(S(t_f, t)B(t, U(t)))) > 0$, *system* (7) *is approximately locally null-controllable.*

Proof. (a) From the definition of approximately locally null-controllability we have that there is a positive δ such that for all $x_0 \in \mathrm{int}(\mathrm{B}_\delta)$ it holds that $0 \in \mathrm{cl}(R(t_f, x_0))$. Then $0 \leq \sigma(x^*(\mathrm{cl}(R(t_f, x_0))))$. Also $0 \leq \sigma(x^*(R(t_f, x_0)))$. Using theorem 2.2 in [**?**], we have

$$\begin{aligned} 0 &\leq \sigma(x^*(R(t_f, x_0))) \\ &= \sigma(x^*(S(t_f, t_0)x_0)) + \sigma(x^*(\int_{t_0}^{t_f} S(t_f, t)B(t, u(t))))dt \\ &= \sigma(x^*(S(t_f, t_0)x_0)) + \int_{t_0}^{t_f} \sigma(x^*(S(t_f, t)B(t, u(t))))dt, \end{aligned}$$

for any $x_0 \in \mathrm{int}(\mathrm{B}_\delta)$ and $x^* \in X^*$. Therefore we can write

$$0 \leq \int_{t_0}^{t_f} \sigma(x^*(S(t_f,t)B(t,u(t))))dt.$$

Since $S(t_f,t)B(t,U(t)) \neq \{0\}$ on a set of positive Lebesgue measure, we see that there exists $x^* \in X^* \setminus \{0\}$ and $E \subset T$ Lebesgue measurable, with $\mu(E) > 0$ such that $0 < \sigma(x^*(S(t_f,t)B(t,U(t))))$, for all $t \in E$.

(b) Choose $x^* \in X^* \setminus \{0\}$. Then choose $E \subset T$ Lebesgue measurable with $\mu(E) > 0$ such that for all $t \in E$ $\sigma(x^*(S(t_f,t)B(t,U(t)))) > 0$. Thus we can define the nonempty multifunction L as

$$E \ni t \rightsquigarrow L(t) := \{u \in U(t) \mid x^*(S(t_f,t)B(t,u)) > 0\}.$$

We consider the following mapping

$$E \times Y \ni (t,u) \mapsto g(t,u) := x^*(S(t_f,t)B(t,u))$$

and remark that it is Carathéodory. Then by theorem 6.5 in [**?**] the multifunction

$$E \ni t \rightsquigarrow H(t) := x^*(S(t_f,t)B(t,U(t)))$$

is weakly measurable, hence graph measurable. Recalling that g is Carathéodory and using corollary 6.3 in [**?**], we have that the set

$$\{(t,u) \mid x^*(S(t_f,t)B(t,u)) > 0\}$$

is measurable. Then the multifunction L is graph measurable since

$$graph(L) = graph(H) \cap \{(t,u) \mid x^*(S(t_f,t)B(t,u)) > 0\}.$$

Using the Aumann selection theorem, we get a measurable selection u_1 from L such that $u_1(t) \in L(t)$, a. e. on E.

Now as we mentioned in (c) of Remarks 2.1 the mapping

$$T \times Y \ni (t,u) \mapsto S(t_f,t)B(t,u)$$

is Carathéodory. U has complete values. Then by theorem 6.5 in [**?**] the multifunction

$$T \ni t \rightsquigarrow S(t_f,t)B(t,U(t))$$

is weakly measurable. Thus it is graph measurable. By hypothesis $0 \in S(t_f, t)B(t, U(t))$, for all $t \in T$. Then by theorem 7.2 in [?], we get a measurable selection $u_2(t) \in U(t)$, $t \in T$, such that

$$0 = S(t_f, t)B(t, u_2(t)), \quad \text{a.e.}$$

The selections u_1 and u_2 are integrable, too. Thus we can define

$$\hat{u} = \chi_E u_1 + \chi_{T \setminus E} u_2 \in S_U^1.$$

Let $\hat{x} \in C(T, X)$ be the (unique) mild solution generated by $\hat{u}$ and starting from the origin, i.e., $x_0 = 0$. Then we have

$$x^*(\hat{x}(t_f)) = \int_{t_0}^{t_f} \sigma(x^*(S(t_f, t)B(t, \hat{u}(t))))dt$$
$$= \int_E \sigma(x^*(S(t_f, t)B(t, u_1(t))))dt > 0.$$

Thus

$$\sigma(x^*(R(t_f, 0))) > 0.$$

Since $x \mapsto \sigma(x^*(R(t_f, x)))$ is continuous, we can find $\delta > 0$ such that for all $x \in \operatorname{int} \mathrm{B}_\delta$ we have $\sigma(x^*(R(t_f, x))) > 0$. Then $0 \in \operatorname{clco} R(t_f, x) = \operatorname{cl} R(t_f, x)$ for all $x \in \operatorname{int} \mathrm{B}_\delta$ and thus system (7) is approximately locally null-controllable.

Now the proof is complete.

Remarks 2.2.

(a) Our theorem 2.1 is related to theorem 2.2 in [?].

(b) In theorem 2.2 in [?] the multifunction U is considered having convex values and being on a weakly compact subset of Y. We need not such an assumption of convexity of U. Regarding the second assumption, we have required instead that U is integrably bounded.

(c) In theorem 2.2 in [?] the Carathéodory mapping B has linear growth. We need not such an assumption.

References

BABEŞ-BOLYAI UNIVERSITY, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, M. KOGĂLNICEANU 1, 3400 CLUJ-NAPOCA, ROMANIA
E-mail address: mmarian@math.ubbcluj.ro