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CONTINUITY OF THE SOLUTION OF A NONLINEAR PDE WITH RESPECT TO THE DOMAIN

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Dedicated to Professor Wolfgang W. Breckner at his 60^{th} *anniversary*

Abstract. In this paper we consider a nonlinear variational problem and study the continuity of a solution with respect to the domain. The topology on the set of domains is the Hausdorff complementary topology. In the end, the continuity is used to prove the existence of a solution for an optimal shape design problem.

1. **Introduction**

A very actual research field, shape optimization deals with problems in which the optimization variable is the shape of a geometric domain. The existence of solutions for such a problem has been studied in many works. For example, optimal shape design problems for PDEs were considered in [8], [6], [1]; for variational inequalities in [8], [1], [4], [5]; for hemivariational inequalities in [2], [3].

An essential point in the study of optimal shape design problems is the choice of the convergence of the domains. In this paper, following [1] we shall consider the Hausdorff complementary topology, also used in [6], [5].

The shape optimization problem that we study is given in a general form and the system is governed by a nonlinear variational equality. This is a more general setting than the one in [1], where the variational problem is linear. After introducing some preliminary notions, we prove the continuity of the solution of the variational equality with respect to the underlying domain. We formulate then a shape optimization problem and prove that it has at least a solution.

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2. Preliminaries

We present here some notions and results used in the paper, following [1].
Let D be a bounded, open, nonempty subset of $\mathbb{R}^N$.
Denote $\mathcal{G}(D) = \{\Omega \subset D \mid \Omega \text{ open }, \Omega \neq \mathbb{R}^N\}$. The Hausdorff complementary metric ρ_H^C is defined by:

$$\rho_H^C(\Omega_1, \Omega_2) = \|d_{C\Omega_2} - d_{C\Omega_1}\|_{C(D)},$$

where the distance function for a set $A \subset \mathbb{R}^N$ is:

$$d_A(x) = \begin{cases} \inf\limits_{y \in A} |y - x|, & A \neq \emptyset \\ +\infty, & A = \emptyset \end{cases}$$

and $C\Omega$ is the complementary set of Ω.
The metric topology induced is complete and the Hausdorff complementary covergence is denoted by $\Omega_n \xrightarrow{H^C} \Omega$.

Theorem 1. *(i) The space $(\mathcal{G}(D), \rho_H^C)$ is a compact metric space.*
*(ii) Let $\{\Omega_n\}$ be a sequence in $\mathcal{G}(D)$, Ω in $\mathcal{G}(D)$ such that $\Omega_n \xrightarrow{H^C} \Omega$. For any compact subset $K \subset \Omega$, there exists $N(K) \in \mathbb{N}$ such that for all $n \geq N(K)$, $K \subset \Omega_n$ (*compactivorous property*) .*

The domains considered in this paper are of a special type, more precisely they satisfy the uniform cone property.
Given $\lambda > 0$, $0 < \omega \leq \pi/2$ and a direction $d \in \mathbb{R}^N$, $|d| = 1$, we denote $C(\lambda, \omega, d)$ the set

$$C(\lambda, \omega, d) = \{y \in \mathbb{R}^N \ : \ \frac{1}{\tan \omega}|P_H(y)| < y \cdot d < \lambda\}$$

where P_H is the orthogonal projection onto the hyperplane H through the origin and orthogonal to the direction d. The translated cone for $x \in \mathbb{R}^N$ is $C_x(\lambda, \omega, d) = x + C(\lambda, \omega, d)$.
Let $\Omega \subset \mathbb{R}^N$ with $\partial\Omega \neq 0$. Ω is said to satisfy the *uniform cone property* if

$$\exists \lambda > 0, \exists \omega > 0, \exists r > 0 \text{ such that } \forall x \in \partial\Omega, \exists d \in \mathbb{R}^N, |d| = 1$$
$$\text{such that } \forall y \in B(x, r) \cap \bar{\Omega} \text{ we have } \ C_y(\lambda, \omega, d) \subset \text{int}\Omega$$

It is proved in [1] that the family of open lipschitzian domains included in D, which satisfy the uniform cone property is compact with respect to the Hausdorff complementary topology. This family will be denoted with $L(D, r, \omega, \lambda)$.

Let Ω be an open subset of D and $\phi \in \mathcal{D}(\otimes)$, the space of infinitely smooth and compactly supported on Ω functions. Denoting by $e_0(\phi)$ the extension by zero of ϕ to D, we have that $e_0(\phi) \in \mathcal{D}(\mathcal{D})$. By definition, $\|\phi\|_{H^1(\Omega)} = \|e_0(\phi)\|_{H^1(D)}$ and e_0 extends by continuity and density to a linear isometric map between two Sobolev spaces, i.e. $e_0 : H_0^1(\Omega) \to H_0^1(D)$. Denote by $H_0^1(\Omega; D)$ the image of $H_0^1(\Omega)$ by e_0.

Theorem 2. *(i) The linear subspace $H_0^1(\Omega; D)$ of $H_0^1(D)$ is closed and isometrically isomorphic to $H_0^1(\Omega)$.*
(ii) If $\psi \in H_0^1(\Omega; D)$ then $\psi|_\Omega \in H_0^1(\Omega)$ and $\forall \alpha$, $|\alpha| \le 1$ $\partial^\alpha \psi = 0$ a.e. in $D \setminus \Omega$.
(iii) If a sequence converges in $H_0^1(\Omega)$- weak then it converges in $L^2(\Omega)$- strong.

3. Main result

Let Ω and D be bounded, open subsets of $\mathbb{R}^N$, $\Omega \subset D$ and consider the variational equality

$$\begin{aligned} &\text{Find } u_\Omega \in H_0^1(\Omega) \text{ such that} \\ &\int_\Omega \{A(x, u_\Omega(x))\nabla u_\Omega(x) \cdot \nabla\phi(x) + a(x, u_\Omega(x))\phi(x)\}dx \\ &= \langle f|_\Omega, \phi\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}, \ \forall \phi \in H_0^1(\Omega) \end{aligned} \tag{1}$$

where $f \in H^{-1}(D)$, $f|_\Omega$ denotes the restriction of f belonging to $H^{-1}(\Omega)$, A and a are functions such that: $A = (a_{ij})_{i,j=1}^N$, $a_{ij} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $a : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, A_i is the i-th row of the matrix A. We suppose that these functions have the following properties:

(P1) a_{ij} and a are measurable with respect to the first variable, $A_i(x, \eta) \cdot \xi$ are continuous with respect to (η, ξ), for a.e. $x \in \mathbb{R}^N$ and for all $i, j = 1, \dots, N$,

(P2) $|a(x, \eta) - a(x, \tilde{\eta})| \le c_1|\eta - \tilde{\eta}|$ and $|a_{ij}(x, \eta) - a_{ij}(x, \tilde{\eta})| \le c_2|\eta - \tilde{\eta}|$ for a.e. $x \in \mathbb{R}^N$ and for all $\eta, \tilde{\eta} \in \mathbb{R}$, with c_1, c_2 positive constants,

(P3) $\sum_{i,j=1}^{N} a_{ij}(x, \eta)\xi_i\xi_j \ge c_3\|\xi\|_N^2$ and $a(x, \eta)\eta \ge c_4|\eta|^2$, for a.e. $x \in \mathbb{R}^N$ and for all $\eta \in \mathbb{R}$, $\xi \in \mathbb{R}^N$,

(P4) $|A_i(x, \eta) \cdot \xi| \le c_5(k_1(x) + |\eta| + \|\xi\|)$, $|a(x, \eta)| \le c_6(k_2(x) + |\eta|)$ for a.e. $x \in \mathbb{R}^N$

and for all $\eta, \tilde{\eta} \in \mathbb{R}$, with $k_1, k_2 \in L^2(D)$ positive functions.

According to [7], pg. 76 we have:

Theorem 3. *In the conditions mentioned above, the variational problem (1) has at least a solution* u_Ω.

Lemma 4. *If* $u_\Omega \in H_0^1(\Omega)$ *is a solution of the variational problem (1), and* $u = e_0(u_\Omega)$, *then*

$$\|u\|_{H_0^1(D)} \leq \alpha \|f\|_{H^{-1}(D)}, \tag{2}$$

with α *a positive constant.*

Proof. $u = e_0(u_\Omega)$ is a solution of the variational problem

$$\begin{aligned} & u \in H_0^1(\Omega; D) \text{ such that} \\ & \int_D \{A(x, u(x))\nabla u(x) \cdot \nabla \phi(x) + a(x, u(x))\phi(x)\}dx \\ & = \langle f, \phi \rangle_{H^{-1}(D) \times H_0^1(D)}, \forall \phi \in H_0^1(\Omega; D) \end{aligned} \tag{3}$$

Using **(P3)**, Hölder and Poincaré inequalities we get:

$$\begin{aligned} \|u\|^2_{H_0^1(D)} & \leq \frac{1}{c_3} \int_D A(x, u(x))\nabla u(x) \cdot \nabla u(x) dx + \frac{1}{c_4} \int_D a(x, u(x))u(x)dx \\ & \leq \alpha \int_D \{A(x, u(x))\nabla u(x) \cdot \nabla u(x) + a(x, u(x))u(x)\}dx \\ & = \alpha \langle f, u \rangle_{H^{-1}(D) \times H_0^1(D)} \leq \alpha \|f\|_{H^{-1}(D)} \|u\|_{H_0^1(D)}. \quad \square \end{aligned}$$

We consider $\{\Omega_n\}$ a sequence of open subsets of D and the corresponding variational equalities:

$$\begin{aligned} & u_{\Omega_n} \in H_0^1(\Omega_n) \text{ such that} \\ & \int_{\Omega_n} \{A(x, u_{\Omega_n}(x))\nabla u_{\Omega_n}(x) \cdot \nabla \phi(x) + a(x, u_{\Omega_n}(x))\phi(x)\}dx \\ & = \langle f|_{\Omega_n}, \phi \rangle_{H^{-1}(\Omega_n) \times H_0^1(\Omega_n)}, \forall \phi \in H_0^1(\Omega_n) \end{aligned} \tag{4}$$

Denoting with $u_n = e_0(u_{\Omega_n})$ the extension by zero to D of u_{Ω_n}, this satisfies

$$\begin{aligned} & u_n \in H_0^1(\Omega_n; D) \text{ such that} \\ & \int_D \{A(x, u_n(x))\nabla u_n(x) \cdot \nabla \phi(x) + a(x, u_n(x))\phi(x)\}dx \\ & = \langle f, \phi \rangle_{H^{-1}(D) \times H_0^1(D)}, \forall \phi \in H_0^1(\Omega_n; D) \end{aligned} \tag{5}$$

It takes place:

Theorem 5. *Let $D \subset \mathbb{R}^N$ be a bounded, open, nonempty domain; $\{\Omega_n\}$ a sequence of open subsets of D with $\Omega_n \stackrel{H^C}{\to} \Omega$. Denote by u_n a solution of (5). Then there exists a subsequence (still denoted u_n) and $u \in H_0^1(D)$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(D)$, $u = 0$ a.e. in $D \setminus \bar{\Omega}$ and*

$$\begin{aligned}&\int_D \{A(x,u(x))\nabla u(x)\cdot\nabla\phi(x) + a(x,u(x))\phi(x)\}dx\\&= \langle f,\phi\rangle_{H^{-1}(D)\times H_0^1(D)},\ \forall \phi \in H_0^1(\Omega; D)\end{aligned} \tag{6}$$

(or equivalently

$$\begin{aligned}&\int_\Omega \{A(x,u(x))\nabla u(x)\cdot\nabla\phi(x) + a(x,u(x))\phi(x)\}dx\\&= \langle f|_\Omega,\phi\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)},\ \forall \phi \in H_0^1(\Omega)\ ).\end{aligned}$$

If, in addition, the domain Ω is locally lipschitzian, then $u \in H_0^1(\Omega; D)$.

Proof. According to the Lemma 4, for each $n \in \mathbb{N}$ we have:

$$\|u_n\|_{H_0^1(D)} \le \alpha\|f\|_{H^{-1}(D)}$$

which implies the existence of a subsequence, still denoted by u_n, and of an element $u \in H_0^1(D)$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(D)$.

Let $\phi \in \mathcal{D}(\Omega)$ and $K = \mathrm{supp}\phi$ a compact subset of Ω. Then, according to Theorem 1 there exists a rank $N(K) > 0$ such that for all $n \ge N(K)$, $K \subset \Omega_n$ and

$$\int_D \{A(x,u_n(x))\nabla u_n(x)\cdot\nabla\phi(x) + a(x,u_n(x))\phi(x)\}dx = \langle f,\phi\rangle_{H^{-1}(D)\times H_0^1(D)}$$

We want to pass to the limit in this equality. We have:

$$\begin{aligned}&|\int_D a(x,u_n(x))\phi(x)dx - \int_D a(x,u(x))\phi(x)dx|\\&\le \int_D |a(x,u_n(x)) - a(x,u(x))||\phi(x)|dx\\&\le c_1\int_D |u_n(x)-u(x)||\phi(x)|dx \le c_1\|u_n-u\|_{L^2(D)}\|\phi\|_{L^2(D)} \to 0,\end{aligned}$$

since the weak convergence in $H_0^1(D)$ implies the strong convergence in $L^2(D)$.

The mappings $x \mapsto A(x,u_n(x))$, $x \mapsto A(x,u(x))$, $x \mapsto A^T(x,u_n(x))$ and $x \mapsto$

$A^T(x,u(x))$ belong to $L^2(D)$:

$$\int_D |a_{ij}(x,u_n(x))|^2dx \le \int_D c_5^2(\bar{k}_1(x)+|u_n(x)|)^2dx$$
$$= c_5^2\int_D\{|\bar{k}_1(x)|^2+|u_n(x)|^2+2|\bar{k}_1(x)||u_n(x)|\}dx$$
$$\le c_5^2\{\int_D |\bar{k}_1(x)|^2dx+\int_D |u_n(x)|^2dx$$
$$+2\big(\int_D |\bar{k}_1(x)|^2dx\big)^{1/2}\big(\int_D |u_n(x)|^2dx\big)^{1/2}\}<\infty,$$

where $\bar{k}_1(x)=k_1(x)+1$.

From $\phi\in\mathcal{D}(\Omega)$ it follows that $\nabla\phi\in L^\infty(D)$ so the mapping $x\mapsto A^T(x,u_n(x))\nabla\phi(x)$ is also in $L^2(D)$ and converges strongly to the mapping $x\mapsto A^T(x,u(x))\nabla\phi(x)$. Indeed,

$$\|A^T(\cdot,u_n(\cdot))\nabla\phi(\cdot)-A^T(\cdot,u(\cdot))\nabla\phi(\cdot)\|^2_{L^2(D)}$$
$$=\int_D \|A^T(x,u_n(x))\nabla\phi(x)-A^T(x,u(x))\nabla\phi(x)\|_N^2dx$$
$$\le\int_D \|A^T(x,u_n(x))-A^T(x,u(x))\|^2_{N^2}\cdot\|\nabla\phi(x)\|_N^2dx$$
$$\le\int_D c^2N^4|u_n(x)-u(x)|^2\|\nabla\phi(x)\|_N^2dx$$
$$\le\|\nabla\phi\|^2_{L^\infty(D)}c^2N^4\int_D |u_n(x)-u(x)|^2dx\to 0.$$

(We used here the fact that $\|A^T(x,\eta)-A^T(x,\tilde{\eta})\|_{N^2}\le cN^2|\eta-\tilde{\eta}|$ which follows immediately from **(P2)**).

We have now the convergences:

$$A^T(\cdot,u_n(\cdot))\nabla\phi(\cdot)\to A^T(\cdot,u(\cdot))\nabla\phi(\cdot) \text{ strongly in } L^2(D)$$
$$\nabla u_n(\cdot)\rightharpoonup\nabla u(\cdot) \text{ weakly in } L^2(D),$$

which implies that

$$\int_D A^T(x,u_n(x))\nabla\phi(x)\cdot\nabla u_n(x)dx\to\int_D A^T(x,u(x))\nabla\phi(x)\cdot\nabla u(x)dx$$

hence

$$\int_D A(x,u_n(x))\nabla u_n(x)\cdot\nabla\phi(x)dx\to\int_D A(x,u(x))\nabla u(x)\cdot\nabla\phi(x)dx.$$

So $u\in H_0^1(D)$ satisfies the variational equality (6) for every $\phi\in\mathcal{D}(\Omega)$. By density this extends to all ϕ in $H_0^1(\Omega;D)$.

The proof of the other statements in the theorem is as in [1]:
$u_n = 0$ almost everywhere in $D \setminus \Omega_n$. Then

$$\int_D \chi_{C\bar{\Omega}_n} |u(x)|^2 = \int_{D \setminus \bar{\Omega}_n} |u_n(x) - u(x)|^2 dx \leq \int_D |u_n(x) - u(x)|^2 dx \to 0$$

and so

$$0 = \liminf_{n \to 0} \int_D \chi_{C\bar{\Omega}_n} |u_n(x) - u(x)|^2 dx \geq \int_D \chi_{C\bar{\Omega}} |u(x)|^2 dx.$$

Therefore $u \in H_0^1(D)$, $u = 0$ a.e. in $D \setminus \bar{\Omega}$. For lipschitzian domains, the trace of u is well defined on $\partial\Omega$. It is zero since u and ∇u are zero a.e. in the locally lipschitz domain $C\bar{\Omega}$ by using the Gauss-Green formula. □

Remark 6. If the matrix function $A = (a_{i,j})_{i,j=1}^N$ is such that $a_{ij} : \mathbb{R}^N \to \mathbb{R}$, $A \in L^\infty(D; \mathcal{L}(\mathbb{R}^\mathcal{N}, \mathbb{R}^\mathcal{N}))$ with $a_{ij} = a_{ji}$, $\alpha I \leq A \leq \beta I$ ($0 < \alpha \leq \beta$ constants) and $a = 0$ then the variational problem (1) becomes a linear one :

$$\int_\Omega A(x) \nabla u_\Omega(x) \cdot \nabla \phi(x) dx = \langle f|_\Omega, \phi \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}, \ \forall \phi \in H_0^1(\Omega).$$

for which the continuity of the solution with respect to the underlying domain is proved in [1], Th. 4.1.,p. 266.

We consider now the shape optimization problem:

$$\begin{aligned} &\text{Find } (\Omega^*, u^*) \in \bigcup_{\Omega \in L(D,r,\omega,\lambda)} \{\Omega \times S(\Omega)\} \text{ such that} \\ &J(\Omega^*, u^*) = \min_{\Omega \in L(D,r,\omega,\lambda)} \min_{v \in S(\Omega)} J(\Omega, v) \end{aligned} \tag{7}$$

We say that the pair (Ω_n, v_n) converges to (Ω, v) if

$$\begin{aligned} &\text{(i) } \Omega_n \stackrel{H^C}{\to} \Omega \text{ and} \\ &\text{(ii) } e_0(v_n) \to e_0(v) \text{ in } L^2(D) \end{aligned} \tag{8}$$

We make the hypothesis (see also [3]) that the cost functional J is lower semicontinuous with respect to the convergence: $(\Omega_n, v_n) \to (\Omega, v)$.

Theorem 7. *In the conditions stated above, the optimization problem (7) admits at least one solution.*

Proof. We shall use the same ideea as in the direct method of the calculus of variations.
Let (Ω_n, u_{Ω_n}) be a minimizing sequence for the problem (7). The family $L(D, r, \omega, \lambda)$ is compact with respect to the Hausdorff complementary topology, so there exists a

subsequence of Ω_n, still denoted by Ω_n, and a set $\Omega \in L(D, r, \omega, \lambda)$ such that $\Omega_n \stackrel{H^C}{\to} \Omega$. Next, since $u_{\Omega_n} \in S(\Omega_n)$ we get, according to Theorem 5, that there exists a subsequence u_{Ω_n} and $u \in H_0^1(\Omega; D)$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(D)$, $u_n \to u$ strongly in $L^2(D)$ $(u_n = e_0(u_{\Omega_n}))$. Also, u satisfies the variational equality (6), which means $u|_\Omega \in S(\Omega)$.

Finally, by the fact that the cost functional J is lower semicontinous, $(\Omega, u|_\Omega)$ is a solution for the optimization problem (7). $\square$

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