STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 4, December 2002

APPPROXIMATION OF BIVARIATE FUNCTIONS BY MEANS OF THE OPERATORS $S_{m,n}^{\alpha,\beta;a,b}$

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Dedicated to Professor D.D. Stancu on his 75^{th} *birthday*

Abstract. By starting from the Steffensen theta operator $\theta^{\alpha,\beta}$, defined at (2.1), one constructs the bivariate operator given at (2.2), which depends on the parameters α, β, a, b. In the case $\beta = b = 0$ one obtains the Stancu operators $S_{m,n}^{\alpha;a}$, investigated anterior in the paper [10]. In the case $\alpha = a = 0$ we get a bivariate operator of Cheney-Sharma. For the remainder of the approximation formula (3.1) we present three representations: (3.2), (3.3) and (3.4). In the final part of the paper we give estimations of the order of approximation of a bivariate function f by means of the operators introduced at (2.2).

1. **Introduction**

It is known that the **omega operators** Ω, considered in 1902 by Jensen [3], include the **shift operator** E^a, defined by $(E^a f)(x) = f(x+a)$, the **central mean operator** μ, defined by

$$(\mu_h f)(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right)+f\left(x-\frac{h}{2}\right)\right]$$

and the **integration operator**.

An operator T which commutes with all shift operators is called a **shift invariant operator**, that is $TE^a = E^a T$.

A special case of an omega operator is represented by the **theta operator** θ, introduced in 1927 in his book [11] by J.F. Steffensen. Such an operator is sometime called **delta operator** and is denoted by Q in the book of F.B. Hildebrand [2],

Received by the editors: 01.09.2002.

2000 *Mathematics Subject Classification.* 41A10, 41A25, 41A36, 41A63, 41A80.

Key words and phrases. Bernstein operators, umbral calculus, remainders, order of approximation.

published in 1956. This last term was used very often by specialists in **umbral calculus**: G.-C. Rota [6], S. Roman [5] and others.

A **theta operator** θ is a **shift-invariant operator** for which θe_1 is a constant different from zero, where $e_1(t) = t$.

Typical examples of theta operators are represented by the **forward, backward** and **central differences** operators $\Delta_h, \nabla_h, \delta_h$, as well as by the **prederivative operator** $D_h = \Delta_h/h$. We consider that D_0 is the derivative operator D.

Another, very interesting example is represented by the **Abel operator** $A_a = DE^a = E^a D$, which in the case of $p_m(x; a) = x(x - ma)^{m-1}$ leads to the formula:

$$A_a p_m(x; a) = mx(x - (m - 1)a)^{m-2}.$$

It is known that a θ operator can be expressed as a power series in the derivative operator.

One can see that: (i) for every theta operator θ we have $\theta c = 0$, where c is a constant; (ii) if p_m is a polynomial of degree m, then θp_m is of degree $m - 1$. This is the reason that the θ operators are called **reductive operators**.

A sequence of polynomials (p_m) is called by I.M. Sheffer [7] and Gian-Carlo Rota [6], as well by his collaborators, the sequence of **basic polynomials** if we have: $p_0(x) = 1$, $p_m(0) = 0$ $(m \geq 1)$, $\theta p_m = mp_{m-1}$. These polynomials were called by Steffensen [12] **poweroids**, considering that they represent an extension of the mathematical notion of power.

It is easy to see that: (i) if (p_m) is a basic sequence of polynomials for a theta operator, then it is a basic sequence; (ii) if (p_m) is a sequence of basic polynomials, then it is a basic sequence for a theta operator.

By induction can be proved that every theta operator has a unique sequence of basic polynomials associated with it.

J.F. Steffensen [12] observed that the property of the polynomial sequence $e_m(x) = x^m$ to be of binomial type, can be extended to any sequence of basic polynomials associated to a theta operator.

Illustrative examples: (i) if θ is the derivative operator D, then $p_m(x) = x^m$; (ii) if θ is the prederivative operator $D_h = \Delta_h/h$, then we obtain the factorial power:

$$p_m(x) = x^{[m,h]} = x(x - h) \dots (x - (m - 1)h).$$

2. Use of the Steffensen theta operator $\theta^{\alpha,\beta}$ for construction the approximating operators $S_{m,n}^{\alpha,\beta;a,b}$

Now let us consider the **theta operator of Steffensen** [12]:

$$\theta^{\alpha,\beta} = \frac{1}{\alpha}[1 - E^{-\alpha}]E^{\beta}, \tag{2.1}$$

where α and β are nonnegative parameters.

In this case the basic polynomials are

$$p_m(x;\alpha,\beta) = p_m^{\alpha,\beta}(x) = x(x+\alpha+m\beta)^{[m-1,-\alpha]} = \frac{x}{x+m\beta}(x+m\beta)^{[m,-\alpha]}.$$

These are polynomials of binomial type.

By using them we can give a generalized Abel-Jensen combinatorial formula

$$(x+y)(x+y+m\beta)^{[m-1,-\alpha]} =$$
$$= \sum_{k=0}^{m} \binom{m}{k} x(x+\alpha+k\beta)^{[k-1,-\alpha]} y(y+\alpha+(m-k)\beta)^{[m-1-k,-\alpha]}.$$

Selecting $y = 1-x$ we can write the identity

$$(1+\alpha+m\beta)^{[m-1,-\alpha]} =$$
$$= \sum_{k=0}^{m} \binom{m}{k} x(x+\alpha+k\beta)^{[k-1,-\alpha]}(1-x)(1-x+\alpha+(m-k)\beta)^{[m-1-k,-\alpha]}.$$

We introduce the polynomials $p_{m,k}^{\alpha,\beta}(x)$, defined by the relation

$$(1+\alpha+m\beta)^{[m-1,-\alpha]} p_{m,k}^{\alpha,\beta}(x) =$$
$$= \sum_{k=0}^{m} \binom{m}{k} x(x+\alpha+k\beta)^{[k-1,-\alpha]}(1-x)(1-x+\alpha+(m-k)\beta)^{[m-1-k,-\alpha]}.$$

Let f be a real-valued bivariate function defined on the square $D = [0,1] \times [0,1]$.

We define the bivariate operator $S_{m,n}^{\alpha,\beta;a,b}$ by means of the formula

$$(S_{m,n}^{\alpha,\beta;a,b} f)(x,y) = \sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}^{\alpha,\beta}(x) q_{n,j}^{a,b}(y) f\left(\frac{i}{m}, \frac{j}{n}\right), \tag{2.2}$$

where

$$(1+a+nb)^{[n-1,-a]} q_{n,j}^{a,b}(y) = \binom{n}{j} y(y+a+jb)^{[j-1,-a]}(1-y)(1-y+a+(n-j)b)^{[n-1-j,-a]}.$$

Now we present two special cases of this operator:

(i) In the case $\beta = b = 0$ we have

$$(S_{m,n}^{\alpha;a}f)(x,y) = \sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}^{\alpha}(x)q_{n,j}^{a}(y)f\left(\frac{i}{m},\frac{j}{n}\right),$$

where

$$p_{m,k}^{\alpha}(x) = \binom{m}{k}x^{k,-\alpha}(1-x)^{[m-k,-\alpha]}/1^{[m,-\alpha]},$$

$$q_{n,j}^{a}(y) = \binom{n}{j}y^{[j,-\alpha]}(1-y)^{[n-j,-a]}/1^{[n,-a]}.$$

The approximation properties of this operator have been studied in the paper [10].

(ii) If $\alpha = a = 0$ we obtain

$$(S_{m,n}f)(x,y;\beta,b) = \sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(x;\beta)q_{n,j}(y;b)f\left(\frac{i}{m},\frac{j}{n}\right),$$

where

$$p_{m,k}(x;\beta) = \frac{\binom{m}{k}x(x+k\beta)^{k-1}(1-x+(m-k)\beta)^{m-k-1}}{(1+m\beta)^{m-1}}$$

and

$$q_{n,j}(y;b) = \frac{\binom{n}{j}y(y+jb)^{j-1}(1-y+(n-j)b)^{n-j-1}}{(1+nb)^{n-1}}.$$

This operator represents **an extension to two variables of the second operator of Cheney-Sharma** [1].

We can see that

$$(S_{m,n}e_{0,0})(x,y) = 1, \quad (S_{m,n}e_{1,0})(x,y) = x,$$

$$(S_{m,n}e_{0,1})(x,y) = y, \quad (S_{m,n}e_{1,1})(x,y) = xy.$$

For $e_{2,0}(x,y) = x^2$ and $e_{0,2}(x,y) = y^2$ we have

$$(S_{m,n}e_{2,0})(x,y) = (S_m e_2)(x),$$

$$(S_{m,n}e_{0,2})(x,y) = (S_n e_2)(y)$$

and we can write [1]:

$$\lim_{m\to\infty}(S_m e_2)(x) = x^2, \quad \lim_{n\to\infty}(S_n e_2)(y) = y^2,$$

uniformly on the interval $[0,1]$.

According to the bivariate criterion of Bohman-Korovkin, we can state

Theorem 2.1. *If $f \in C(D)$ and $\alpha = \alpha(m) \to 0$, $m\beta(m) \to 0$ for $m \to \infty$, while $b = b(n) \to 0$ and $n\beta(n) \to 0$ when $n \to \infty$, then we have*

$$\lim_{m,n\to\infty}(S_{m,n}f)(x,y) = f(x,y),$$

uniformly on the square D.

3. Evaluation of the remainder

Since the approximation formula

$$f(x,y) = (S^{\alpha,\beta;a,b}_{m,n}f)(x,y) + (R^{\alpha,\beta;a,b}_{m,n}f)(x,y) \tag{3.1}$$

has the degree of exactness (1,1), by applying an extension of the Peano theorem (see [8]) we are able to find an integral representation of the remainder.

We now formulate

Theorem 3.1. *If $f \in C^{2,2}(D)$, then we can give the following integral representation for the remainder of formula (3.1):*

$$(R^{\alpha,\beta;a,b}_{m,n}f)(x,y) = \tag{3.2}$$

$$= \int_0^1 G^{\alpha,\beta}_m(t;x)f^{(2,0)}(t,y)dt + \int_0^1 H^{a,b}_n(z,y)f^{(0,2)}(x,z)dz -$$

$$- \int_0^1\int_0^1 G^{\alpha,\beta}_m(t;x)H^{a,b}_n(z,y)f^{(2,2)}(t,z)dtdz,$$

where

$$G^{\alpha,\beta}_m(t,x) = (R^{\alpha,\beta;a,b}_{m,n}\varphi_x)(t),$$

$$H^{a,b}_n(z,y) = (R^{\alpha,\beta;a,b}_{m,n}\psi_y)(z),$$

with

$$\varphi_x(t) = \frac{1}{2}[x - t + |x-t|], \quad \psi_y(z) = \frac{1}{2}[y - z + |y-z|]$$

and the use of the notation

$$f^{(n,s)}(u,v) = \frac{\partial^{r+s}f(u,v)}{\partial u^r\partial v^s} \quad (r,s = 0,1,2).$$

Proof. Formula (3.2) can be obtained if we use a representation of Peano-Milne type, given in the paper [8], for the remainder of a bivariate linear approximation formula having a certain degree of exactness.

If we assume that $x \in \left[\frac{r-1}{m}, \frac{r}{m}\right]$, we can give for the Peano kernel $G_m^{\alpha,\beta}(t,x)$ the following expression

$$G_m^{\alpha,\beta}(t;x) = \begin{cases} -\sum_{k=0}^{i=1} p_{m,k}^{\alpha,\beta}(x)\left(t-\frac{k}{m}\right) & \text{if} \quad t \in \left[\frac{i-1}{m}, \frac{i}{m}\right] \\ & (1 \le i \le r-1) \\ -\sum_{k=0}^{r-1} p_{m,k}^{\alpha,\beta}(x)\left(t-\frac{k}{m}\right) & \text{if} \quad t \in \left[\frac{r-1}{m}, x\right] \\ -\sum_{k=}^{m} p_{m,k}^{\alpha,\beta}(x)\left(\frac{k}{m}-t\right) & \text{if} \quad t \in \left[x, \frac{r}{m}\right] \\ -\sum_{k=i}^{m} p_{m,k}^{\alpha,\beta}\left(\frac{k}{m}-t\right) & \text{if} \quad t \in \left[\frac{i-1}{m}, \frac{i}{m}\right] \\ & (r \le i \le m) \end{cases}$$

The dual Peano kernel $H_n^{a,b}(z,y)$ has a similar expression.

If we take into account that on the square D we have $G_m^{\alpha,\beta}(t,x) \le 0$ and $H_n^{a,b}(z,y) \le 0$, we can apply the first law of the mean to the integrals and we find that

$$(R_{m,n}^{\alpha,\beta;a,b}f)(x,y) =$$
$$= f^{(2,0)}(\xi,y)\int_0^1 G_m^{\alpha,\beta}(t,x)dt + f^{(0,2)}(x,\eta)\int_0^1 H_n^{a,b}(z,y)dz-$$
$$-f^{(2,2)}(\xi,\eta)\left[\int_0^1 G_m^{\alpha,\beta}(t,x)dt\right]\left[\int_0^1 H_n^{a,b}(z,y)dz\right],$$

where ξ and η are certain points from the interval $(0,1)$.

It is easy to see that we have

$$\int_0^1 G_m^{\alpha,\beta}(t,x)dt = \frac{1}{2}(R_m^{\alpha,\beta}e_{2,0})(x),$$

$$\int_0^1 H_n^{a,b}(z,y)dz = \frac{1}{2}(R_n^{a,b}e_{0,2})(y),$$

where $R_m^{\alpha,\beta}$ and $R_n^{a,b}$ are the univariate remainders:

$$R_m^{\alpha,\beta} = I - S_m^{\alpha,\beta}, \quad R_n^{a,b} = I - S_n^{a,b}.$$

Now we can state the following

Corollary 3.1. *If $f \in C^{2,2}(D)$, then the remainder of the approximation formula (3.1) can be represented under the following Cauchy form*

$$(R_{m,n}^{\alpha,\beta;a,b}f)(x,y) = \tag{3.3}$$

$$= \frac{1}{2}(R_m^{\alpha,\beta}e_2)(x)f^{(2,0)}(\xi,y) + \frac{1}{2}(R_n^{a,b}e_2)f^{(0,2)}(x,\eta) -$$

$$-\frac{1}{4}(R_m^{\alpha,\beta}e_2)(x)(R_n^{a,b}e_2)(y)f^{(2,2)}(\xi,\eta).$$

Because $(S_m^{\alpha,\beta}f)(x)$ and $(S_n^{a,b}f)(y)$ are interpolatory at both sides of the interval $[0,1]$, we can conclude that $(R_m^{\alpha,\beta}e_2)(x)$ contains the factor $x(x-1)$, while $(R_n^{a,b}e_2)(y)$ has the factor $y(y-1)$.

Since $R_m^{\alpha,\beta}e_0 = 0$, $R_n^{a,b}e_0 = 0$ and the remainder is different from zero for any convex function f of the first order, we can apply a criterion of T. Popoviciu [4] and we find that the remainder is of simple form. Consequently we can state the following

Theorem 3.2. *If the second-order divided differences of the function f are bounded on the square D, we can give an expression of the remainder of the formula (3.1) in terms of divided differences*

$$(R_{m,n}^{\alpha,\beta;a,b}f)(x,y) = (R_m^{\alpha,\beta}e_{2,0})(x)[x_{m,1},x_{m,2},x_{m,3};f(t,y)] =$$

$$+(R_n^{a,b}e_{0,2})(y)[y_{n,1},y_{n,2},y_{n,3};f(x,z)] -$$

$$-(R_m^{\alpha,\beta}e_{2,0})(x)(R_n^{a,b}e_{0,2})(y)\begin{bmatrix} x_{m,1},x_{m,2},x_{m,3} & ;f(t,z) \\ y_{n,1},y_{n,2},y_{n,3} & \end{bmatrix}, \tag{3.4}$$

where $x_{m,1},x_{m,2},x_{m,3},y_{n,1},y_{n,2},y_{n,3}$ are certain points in the interval $[0,1]$.

If we apply the mean-value theorem to the divided differences, we arrive at the Corollary 3.1.

4. Estimation of the order of approximation

We will use the **bivariate modulus of continuity**

$$\omega(f;\delta_1,\delta_2) = \sup\{|f(x,y)-f(x',y')| : \ |x-x'| \le \delta_1, \ |y-y'| \le \delta_2\},$$

where (x,y) and (x',y') are points of the square D and $\delta_1,\delta_2 \in \mathbb{R}_+$.

Because the constants are reproduced by our operator and $p_{m,k}^{\alpha,\beta}(x) \geq 0$, $q_{n,j}^{a,b}(y) \geq 0$, when $x, y \in [0,1]$, we can write

$$|f(x,y) - (S_{m,n}^{\alpha,\beta;a,b}f)(x,y)| \leq$$
$$\leq \sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}^{\alpha,\beta}(x) q_{n,j}^{a,b}(y) \left| f(x,y) - f\left(\frac{k}{m}, \frac{j}{n}\right)\right|.$$

By using a basic property of the modulus of continuity, we can write

$$|f(x,y) - (S_{m,n}^{\alpha,\beta;a,b}f)(x,y)| \leq$$
$$\leq \left[1 + \frac{1}{\delta_1^2}\sum_{k=0}^{m} p_{m,k}^{\alpha,\beta}(x)\left(x - \frac{k}{m}\right)^2 + \frac{1}{\delta_2^2}\sum_{j=0}^{n} q_{n,j}^{a,b}(y)\left(y - \frac{j}{n}\right)^2\right]\omega(f;\delta_1,\delta_2).$$

Since our partial operators are interpolatory in 0 and 1, we can write

$$\sum_{k=0}^{m} p_{m,k}^{\alpha,\beta}(x)\left(x - \frac{k}{m}\right)^2 = (S_m^{\alpha,\beta}e_2)(x) - x^2 = -(R_m^{\alpha,\beta}e_2)(x) = \frac{x(1-x)}{m}A_m^{\alpha,\beta}.$$

By selecting

$$\delta_1 = c\sqrt{\frac{x(1-x)}{m}}, \quad \delta_2 = d\sqrt{\frac{y(1-y)}{n}} \quad (c > 0,\ d > 0),$$

we get

$$|f(x,y) - (S_{m,n}^{\alpha,\beta;a,b}f)(x,y)| \leq$$
$$\leq \left[1 + \frac{1}{c^2}A_m^{\alpha,\beta} + \frac{1}{a^2}B_n^{a,b}\right]\omega\left(f; c\sqrt{\frac{x(1-x)}{m}}, d\sqrt{\frac{y(1-y)}{n}}\right).$$

If we choose $c = d = 2$ and take into consideration that $t(1-t) \leq \frac{1}{4}$ on $[0,1]$, we can state

Theorem 4.1. *The order of approximation of the function $f \in C(D)$ is evaluated by the following inequality*

$$\|f - S_{m,n}^{\alpha,\beta;a,b}f\| \leq \left[1 + \frac{1}{4}(A_m^{\alpha,\beta} + B_n^{a,b})\right]\omega\left(f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\right),$$

where $A_m^{\alpha,\beta} = o\left(\frac{1}{m}\right)$, $B_n^{a,b} = o\left(\frac{1}{n}\right)$.

In the particular case $\alpha = \beta = a = b = 0$, we obtain the inequality

$$\|f - B_{m,n}f\| \leq \frac{3}{2}\omega\left(f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\right),$$

corresponding to the approximation by the bidimensional Bernstein polynomial $B_{m,n}$.

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