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# ITERATES OF STANCU OPERATORS, VIA CONTRACTION PRINCIPLE

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*Dedicated to Professor D.D. Stancu on his $75^{th}$ birthday*

**Abstract**. In this note we prove that some Stancu operators are weakly Picard operators.

Let $\alpha, \beta \in R$, $0 \leq \alpha \leq \beta$ and let $n \in N^*$. We consider the Stancu operators ([7], [2])

$$P_{n,\alpha,\beta} : C[0,1] \to C[0,1]$$
$$f \mapsto P_{n,\alpha,\beta}(f)$$

where

$$P_{n,\alpha,\beta}(f)(x) := \sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right)\binom{n}{k} x^k (1-x)^{n-k}. \tag{1}$$

Let $P^m_{n,\alpha,\beta}$ be the $m^{th}$ iterate of the operator $P_{n,\alpha,\beta}$. We have

**Theorem 1.** *Let $n \in N^*$ and $\beta > 0$. Then for all $f \in C[0,1]$,*

$$P^m_{n,0,\beta}(f)(x) \to f(0) \text{ as } m \to \infty,$$

*uniformly with respect to $x \in \left[0, \frac{n}{n+\beta}\right]$.*

**Proof.** Consider the Banach space $\left(C\left[0, \frac{n}{n+\beta}\right], \|\cdot\|_C\right)$ where $\|\cdot\|_C$ is the Chebyshev norm. Let

$$X_\gamma := \left\{ f \in C\left[0, \frac{n}{n+\beta}\right] \mid f(0) = \gamma \right\}, \quad \gamma \in R.$$

We remark that

(a) $X_\gamma$ is a closed subset of $C\left[0, \frac{n}{n+\beta}\right]$, $\gamma \in R$;

(b) $X_\gamma$ is an invariant subset of $P_{n,0,\beta}$ for all $\beta > 0$, $n \in N^*$, $\gamma \in R$;

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(c) $C\left[0,\dfrac{n}{n+\beta}\right]=\bigcup_{\gamma\in R}X_\gamma$ is a partition of $C\left[0,\dfrac{n}{n+\beta}\right]$.

Now we prove that

$$P_{n,0,\beta}:X_\gamma\to X_\gamma$$

is a contraction, for all $n\in N^*$, $\beta>0$ and $\gamma\in R$.

Let $f,g\in X_\gamma$. From (1) we have

$$|P_{n,0,\beta}(f)(x)-P_{n,0,\beta}(g)(x)|=|P_{n,0,\beta}(f-g)(x)|\le$$
$$\le\left(\sum_{k=1}^{n}\binom{n}{k}x^k(1-x)^{n-k}\right)\|f-g\|_C=$$
$$=(1-(1-x)^n)\|f-g\|_C\le\left(1-\left(1-\frac{n}{n+\beta}\right)^n\right)\|f-g\|_C.$$

From this we have that

$$\|P_{n,0,\beta}(f)-P_{n,0,\beta}(g)\|_C\le\left(1-\left(1-\frac{n}{n+\beta}\right)^n\right)\|f-g\|_C,$$

for all $f,g\in X_\gamma$.

We remark that $1-\left(1-\dfrac{n}{n+\beta}\right)^n<1$.

On the other hand the constant function $\gamma\in X_\gamma$ and is a fixed point of $P_{n,0,\beta}$.

Let $f\in C\left[0,\dfrac{n}{n+\beta}\right]$. Then $f\in X_{f(0)}$ and from the contraction principle ([5]) it follows that

$$P^m_{n,0,\beta}(f)(x)\to f(0)\text{ as }m\to\infty.$$

**Theorem 2.** *Let $n\in N^*$ and $\alpha>0$. Then for all $f\in C[0,1]$,*

$$P^m_{n,\alpha,\alpha}(f)(x)\to f(1)\ \textit{as}\ m\to\infty,$$

*uniformly with respect to* $x\in\left[\dfrac{\alpha}{n+\alpha},1\right]$.

**Proof.** Let $X_\gamma:=\left\{f\in C\left[\dfrac{\alpha}{n+\alpha},1\right]\mid f(1)=\gamma\right\}$, $\gamma\in R$. Then

(a) $X_\gamma$ is a closed subset of $C\left[\dfrac{\alpha}{n+\alpha},1\right]$, for all $\gamma\in R$;

(b) $X_\gamma$ is an invariant subset of the operator $P_{n,\alpha,\alpha}$, for all $\gamma\in R$, $\alpha>0$ and $n\in N^*$;

(c) $C\left[\dfrac{\alpha}{n+\alpha},1\right]=\bigcup_{\gamma\in R}X_\gamma$ is a partition of $C\left[\dfrac{\alpha}{n+\alpha},1\right]$.

Let us prove that

$$P_{n,\alpha,\alpha}|_{X_\gamma}:X_\gamma\to X_\gamma$$

is a contraction, for all $n \in N^*$, $\alpha > 0$ and $\gamma \in R$.

Let $f, g \in X_\gamma$. From (1) we have

$$\|P_{n,\alpha,\alpha}(f) - P_{n,\alpha,\alpha}(g)\|_C \le \left(1 - \left(\frac{\alpha}{n+\alpha}\right)^n\right) \|f - g\|_C.$$

On the other hand the constant function $\gamma$ is a fixed point of $P_{n,\alpha,\alpha}$ and $\gamma \in X_\gamma$.

Now the proof follows from the contraction principle.

**Remark 1.** For the case $\alpha = \beta = 0$, see [4] and [6].

**Remark 2.** Let $(X, d)$ be a complete metric space. By definition an operator $A : X \to X$ is weakly Picard operator (briefly, WPO) if the sequences $(A^m(x))_{m \in N}$ converges, for all $x \in X$, and the limit (which may depend on $x$) is a fixed point of $A$.

For an WPO we consider the operator $A^\infty$ defined by

$$A^\infty : X \to X, \quad A^\infty(x) := \lim_{m \to \infty} A^m(x).$$

In the terms of WPOs we can formulate the Theorem 1 and 2 as follow

**Theorem 1'.** *Let $n \in N^*$ and $\beta > 0$. Then the Stancu operators $P_{n,0,\beta}$ are WPOs on $C\left[0, \frac{n}{n+\beta}\right]$.*

**Theorem 2'.** *Let $n \in N^*$ and $\alpha > 0$. Then the Stancu operators $P_{n,\alpha,\alpha}$ are WPOs on $C\left[\frac{\alpha}{n+\alpha}, 1\right]$.*

**Remark 3.** The applications of the contraction principle to study the iterations of other approximation operators ([1]-[3]) will be presented elsewhere.

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