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CRITICAL SETS OF 1-DIMENSIONAL MANIFOLDS

LIANA ŢOPAN

Abstract. In this paper we give characterizations for the critical sets of the 1-dimensional manifolds. Given a non-empty set $K \subset M$, with M a smooth manifold of dimension 1, is K the set of critical points for some smooth function $f : M \to \mathbb{R}$?

1. **Introduction**

Let M be a smooth 1-dimensional manifold and $f : M \to \mathbb{R}$ a smooth function. The point $p \in M$ is a *critical point* of f if, for some chart (U, φ) around p, $\varphi(p)$ is a critical point of the function $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$, i.e. $\text{rang}_{\varphi(p)} f \circ \varphi^{-1} = 0$, or $(f \circ \varphi^{-1})'(\varphi(p)) = 0$. Otherwise, p will be a *regular* point of f. The set of all critical points of f is called the *critical set* of f and will be denoted by $C(f)$. The number $y_0 \in \mathbb{R}$ is a *critical value* of f if it is the image of a critical point and a *regular value* if it is the image of a regular point. The set of critical values of f is called the *bifurcation set* of f and is denoted by $B(f)$. A set $C \subset M$ is called *critical* if it is the critical set of some smooth function $f : C \to \mathbb{R}$; $C = C(f)$. C is *properly critical* if f can be chosen to be proper.

If $M = \mathbb{R}$, the atlas which gives the structure of M has one single chart $(\mathbb{R}, 1_{\mathbb{R}})$. In this case, $x \in C(f)$ if and only if $f'(x) = 0$. The following theorem [To-An] characterizes the critical sets of $\mathbb{R}$.

Theorem 1.1. *$C \subset \mathbb{R}$ is critical if and only if C is closed.*

It follows that any finite union of closed bounded intervals (some of them might be degenerated to a point), together with two closed unbounded intervals, one

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of them to $-\infty$ and the other to $+\infty$, is a critical set. Also, any Cantor (real) set, beeing closed, will be critical.

For the case $M = \mathbb{R}$, there are no other requirements for the set C to be critical, except to be closed. This is, in fact, the minimal condition for a set to be critical (it is easy to see that any critical set is closed). If we impose some supplementary conditions on C, it will become properly critical.

Theorem 1.2. *Let C be a subset of $\mathbb{R}$. If C is compact, C is properly critical.*

Proof. C being compact, it is closed, so critical. C is bounded, and there is some $r > 0$ with $C \subset (-r, r)$. Choose $R > r$. Let $g : \mathbb{R} \to \mathbb{R}$ be a smooth positive function which satisfies

1. $g(x) = 1, \quad \forall\, x \in (-r, r)$
2. $g(x) = 0, \quad \forall\, x \in (-\infty, -R) \cup (R, +\infty)$
3. $0 \le g(x) \le 1, \quad \forall\, x \in \mathbb{R}$. (see [To-An]).

A theorem of Whitney provides that any closed subset of $\mathbb{R}$ is the set of the zeros of a smooth positive real function (see [An-To]) and let $f : \mathbb{R} \to \mathbb{R}$ have this property : $C = f^{-1}(0)$. Define $h : \mathbb{R} \to \mathbb{R}$, by

$$h(x) = f(x)g(x) + e^{|x|}(1 - g(x)).$$

h is smooth on $\mathbb{R} \setminus \{0\}$. For $x \in (-r, r)$, since $g(x) = 1$, then $h(x) = f(x)$ and h is smooth on $(-r, r)$, which is an open neighborhood of 0. It follows that h is smooth on the entire $\mathbb{R}$.

It is easy to verify that $h^{-1}(0) = C$. For $x_0 \in C$, since $x_0 \in (-r, r)$, then $g(x_0) = 1$ and $h(x_0) = f(x_0) = 0$. For $h(x_0) = 0$, since $f(x) \ge 0$, $e^{|x|} > 0$ and $0 \le g(x) \le 1$ for all x, then $f(x_0)g(x_0) = e^{|x_0|}(1 - g(x_0)) = 0$, so $f(x_0) = 0$ and $g(x_0) = 1$, which means that $x_0 \in f^{-1}(0) = C$.

Let $H : \mathbb{R} \to \mathbb{R}$ be the function given by $H(x) = \int\limits_0^x h(t)dt$. Obviously, $C(H) = C$. To prove that H is a proper function, it is enough to verify that $|H(x)| \to \infty$ as $|x| \to \infty$ (see [Ra]).

For $x > R$, we have

$$H(x) = \int_0^x h(t)dt = \int_0^R h(t)dt + \int_R^x h(t)dt = \int_0^R h(t)dt + \int_R^x e^t dt =$$

$$= \int_0^R h(t)dt + e^x - e^R = e^x + \int_0^R h(t)dt - e^R$$

so $\lim\limits_{x\to\infty} H(x) = \infty$.

For $x < -R$, we have

$$H(x) = -\int_x^0 h(t)dt = -\int_x^{-R} h(t)dt - \int_{-R}^0 h(t)dt =$$

$$= -\int_{-R}^0 h(t)dt - \int_{-R}^0 e^{-t}dt = -\int_{-R}^0 h(t)dt + e^R - e^{-x}$$

so $\lim\limits_{x\to-\infty} H(x) = -\infty$.

It follows that C is the critical set of the smooth and proper function H, so C is properly critical. □

The converse of the above theorem is not true. There are smooth proper functions $f : \mathbb{R} \to \mathbb{R}$, whose critical sets are not compact. For example, $f(x) = x + \sin x$, whose critical set is $C(f) = \{(2k+1)|\ k \in \mathbb{Z}\}$, discrete and unbounded in $\mathbb{R}$, so non-compact.

2. **Critical Sets on 1-Dimensional Manifolds**

Using the characterization of the connected and compact 1-dimensional manifolds, it follows that it is enough to study the critical sets of the interval $[0,1]$ on the real axis and of the circle S^1 on the plane.

Let M be a smooth 1-dimensional manifold, connected and compact (with or without boundary). According to a theorem of Whitney , M can be properly embedded in $\mathbb{R}^3$ (i.e. there exists an injective and proper immersion $i : M \hookrightarrow \mathbb{R}^3$). Also, there exists $f : M \to \mathbb{R}$ smooth, which is a Morse function (f is said to be a *Morse function* if its critical points are all non-degenerated. The critical points of a Morse function are, also, isolated in M).

Let $S = C(f) \cup \partial M$. As M is of dimension 1, ∂M will be either a smooth compact 0-manifold without boundary, or the empty set. Anyway, ∂M will be at the most a finite union of points. Also, from the compactness of M it follows that $C(f)$ is finite, too, $C(f)$ beeing a discrete subset of a compact. So S is finite and $M \setminus S$ has a finite number of components $L_1,\ldots, L_N$, which are smooth 1-dimensional manifolds.

Proposition 2.1. *f is a diffeomorphism between each L_i and an open interval of $\mathbb{R}$.*

Proof. Let L be one of the manifolds L_i. For all $x \in L$, we have $(df)_x = (df_{|L})_x \neq 0$, so f is a local diffeomorphism on L. Since L is connected, it follows that $f(L)$ is a connected open set. But $f(L)$ is contained in the compact $f(M)$, so $f(L)$ is an open interval (a, b).

We prove now that f is injective on L, and then $f_{|L}$ will be a diffeomorphism. Let $p \in L$ and $c = f(p) \in (a, b)$. Let Q be the set of all points $q \in L$ with the property that there is an arc $\gamma : [c, d] \to L$ joining q and p, $\gamma(c) = p$, $\gamma(d) = q$ and $(f \circ \gamma)(t) = t$, for all $t \in [c, d]$. Since $p \in Q$, then Q is non-empty.

Q is an open set of L: Let $q \in Q$. There is an arc $\gamma : [c, d] \to L$ such that $\gamma(c) = p$, $\gamma(d) = q$ and $(f \circ \gamma)(s) = s$, for s in the interval $[c, d]$. But f beeing a local diffeomorphism in q, there exists a neighborhood V_q of q for which $f_{|V_q} : V_q \to f(V_q)$ is a diffeomorphism. We may choose V_q to be an open connected subset of M. Then $f(V_q) = (c', c'')$, with $a < c' < d < c'' < b$. It follows that γ and $(f_{|V_q})^{-1}$ coincide on

$(c', d]$ and γ can be extended on $[d, c'')$ such that it coincides with $(f_{|V_q})^{-1}$. It follows that any point of V_q can be joined to p, so $V_q \subset Q$ and Q is open in L.

Q is closed in L: It is enough to show that $L \setminus Q$ is open. Let $l \in L \setminus Q$. Then l cannot be joined to p with the requiered conditions. As before, there exists a neighborhood V_l of l with $f_{|V_l} : V_l \to f(V_l)$ diffeomorphism, V_l open and connected and $f(V_l) = (c', c'')$. Suppose there exists a point $q \in V_l$ which can be joined to p. Take $V_q \subset Q$ a neighborhood of q. Every point in $V_l \cap V_q$ can be joined to p, because of V_q and, the same time, cannot be joined to p, beeing on V_l. So, in fact, no point of V_l can be joined to q, which means that $V_l \subset L \setminus Q$, and $L \setminus Q$ is open.

Since L is connected, then $Q = L$. Let $p \neq q$, $p, q \in L$. We showed that there is an arc $\gamma : [c, d] \to L$, with $\gamma(c) = p$, $\gamma(d) = q$ and $(f \circ \gamma)(t) = t$, for all $t \in [c, d]$. We have:

$$f(p) = f(\gamma(c)) = (f \circ \gamma)(c) = c \quad \text{and}$$
$$f(q) = f(\gamma(d)) = (f \circ \gamma)(d) = d,$$

so $f(p) \neq f(q)$, which shows that f is non-injective, so f is a diffeomorphism between L and the open interval (a, b). $\square$

Since every L_i is diffeomorphic to an open interval, then $\overline{L}_i \setminus L_i$ has at the most two points, $\forall\ i = \overline{1, N}$. We can suppose that for all $i = \overline{1, N}$, $\overline{L}_i \setminus L_i$ has exactly two points. Indeed, since L_i is diffeomorphic to an open interval, then $\overline{L}_i \setminus L_i$ has at least one point, and if $\overline{L}_i \setminus L_i$ has exactly one point, it could be only for the case when $N = 1$ and $M = S^1$.

A point $p \in S$ is either a point of the boundary of M, or the intersection point of the boundaries of two sets $\overline{L}_i$ and $\overline{L}_j$. It cannot be the intersection point of three sets $\overline{L}_i$, $\overline{L}_j$ and $\overline{L}_k$, since M is 1-dimensional and the situation below cannot happen.

We call $L_1, \ldots, L_k$ a *chain* if for all $j = \overline{1, k-1}$, $\overline{L}_j$ and $\overline{L}_{j+1}$ have exactly one single intersection point p_j (which belongs to both boudaries). Denote by p_0 the other boundary point of L_1 and by p_k the other boundary point of L_k. Since we have a finite number of L_i, there is a maximal chain, to which we cannot add an other L_i.

Proposition 2.2. *If $L_1, \ldots, L_k$ is a maximal chain, it contains all L_i, $i = \overline{1, N}$. If $\overline{L}_0$ and $\overline{L}_k$ have an intersection point (which will belong to both boundaries), then M is diffeomorphic to a circle. Otherwise, M is diffeomorphic to a closed interval of $\mathbb{R}$.*

Proof. Let us suppose that there exists some L_i which does not belong to the maximal chain. We denote it by L. $\overline{L}$ cannot contain p_0 or p_k, since the chain cannot be extended. $\overline{L}$ contains none of the points p_i, $i = \overline{1, k-1}$, since L_i, L_{i+1} and L would have a common boundary point. It follows that $\overline{L}$ does not intersect $\bigcup_{i=1}^{k} \overline{L}_i$, which is a contradiction to the connectivity of M.

We prove now the second part of the proposition. We construct the requiered diffeomorphisms by using the following lemma:

Lemma 2.3. *Let $g : [a, b] \to \mathbb{R}$ be continuous, smooth on $[a, b] \setminus \{c\}$ and such that $g' > 0$, for all $x \in [a, b] \setminus \{c\}$. Then there exists a smooth map $\breve{g} : [a, b] \to \mathbb{R}$ which agrees to g in a neighborhood of the points a and b and whose derivative is positive on $[a, b]$.*

Sketch of the proof: Let g be a smooth non-negative function, which vanishes outside (a, b), is equal to 1 in a neighbourhood of c and satisfies $\int_a^b g(t)dt = 1$. Define

$\tilde{g} : [a, b] \to \mathbb{R}$, by

$$\tilde{g}(x) = g(a) + \int_a^x [kg(t) + g'(t)(1 - g(t))]dt,$$

with

$$k = g(b) - g(a) - \int_a^b g'(t)(1 - g(t))dt$$

a strictly positive constant. □

The restriction of f to any L_i is a diffeomorphism. The monotony of f could change when f passes through a boundary point of $\overline{L}_i$. To avoid this inconvenient, we use a technical trick. Let $f(p_j) = a_j$. Then $f_{|L_j}$ is a diffeomorphism between L_j and the interval (a_{j-1}, a_j) (or (a_j, a_{j-1})). For each $j = \overline{1, k}$, choose an affine map $\tau_j : \mathbb{R} \to \mathbb{R}$ such that $\tau_j(a_{j-1}) = j - 1$ and $\tau_j(a_j) = j$ (the map τ_j is of the form $t \to \alpha t + \beta$, $\alpha, \beta \in \mathbb{R}$). Let $f_j : \overline{L}_j \to [j - 1, j]$ be the maps given by $f_j = \tau_j \circ f$.

If $a_0 \neq a_k$, the maps f_j will agree on every common point of their domains. We may construct the map $F : M \to [0, k]$, having the following properties:

1. $F_{|\overline{L}_j} = f_j$
2. F is continuous on M
3. F is a diffeomorphism on $M \setminus \{p_1, \ldots, p_{k-1}\}$

By using Lemma 2.3, f can be chosen to be a diffeomorphism on M.

If $a_0 = a_k$, let $g_j = \exp[i\frac{2\pi}{k} f_j]$. We may define now $G : M \to S^1$, such that:

1. $G_{|\overline{L}_j} = g_j$
2. G is continuous on M
3. G is a diffeomorphism on $M \setminus \{p_1, \ldots, p_{k-1}\}$

Again, G can be made to be a global diffeomorphism. □

We obtained

Theorem 2.4. (the classification of connected compact 1-manifolds) *Any smooth connected and compact* 1*-dimensional manifold is diffeomorphic either to* S^1*, or to the interval* $[0, 1]$.

The last theorem provides that it is enough to find the critical sets of S^1 and of $[0, 1]$.

Theorem 2.5. *Let* $K \subset [0,1]$*. Then* K *is critical in* $[0,1]$ *if and only if* K *is closed in* $[0,1]$.

Proof. Any critical set is closed. Conversely, let K be a closed subset of $[0,1]$. Since $[0,1]$ is closed in $\mathbb{R}$, then K is closed in $\mathbb{R}$. According to Theorem 1.1, there is a smooth function $f : \mathbb{R} \to \mathbb{R}$ with $C(f) = K$. Let $g : [0,1] \to \mathbb{R}$, $g = f_{|[0,1]}$. g is smooth and $C(g) = K$. $\square$

Theorem 2.6. *Let* $K \subset S^1$*. Then* K *is critical in* S^1 *if and only if* K *is closed in* S^1.

Proof. If K is critical, K is closed. Conversely, let K be a closed subset of S^1. Suppose that $K \neq S^1$ (S^1 is the critical set of any constant function defined on S^1). K is a compact subset of the plane and the only component of its complement is multiply connected. Using the characterisation of the critical sets of the plane given by Norton and Pugh [No-Pu], it follows that K is the critical set of a smooth map $f : \mathbb{R}^2 \to \mathbb{R}$. $C(f) = K$. Then K will be the critical set of $f_{|S^2} : S^2 \to \mathbb{R}$. $\square$

References

[Gu-Po] Guillemin V., Pollack A., Differential Topology, Prentice Hall, Inc., Englewood Cliffs New-Jersey, 1977.

[No-Pu] Norton A., Pugh Ch., Critical Sets in the Plane, Michigan Journal of Math., 38, 1991, 441-459.

[Ra] Rădulescu S., Rădulescu M., Teoreme şi probleme de analiză matematică, E.D.P., Bucureşti (1982), 104.

[To-An] Ţopan L., Andrica D., Caracterizarea mulţimilor critice pe axa reală, Gazeta Matematică seria A, nr.2/2001, 73-79.

"Babeş-Bolyai" University, Cluj-Napoca