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ON SOME Ω-PURE EXACT SEQUENCES OF MODULES

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Abstract. Let R be an associative ring with non-zero identity. We shall consider a family Ω of left R-modules of the form R/Rr^n, where $r \in R$ and $n \geq 1$ is a natural number depending on r such that $r^n \neq 0$ for each $r \neq 0$. We shall characterize Ω-pure exact sequences of right R-modules and absolutely Ω-pure right R-modules. We shall also establish the structure of Ω-pure-projective right R-modules.

1. **Introduction**

In this paper we denote by R an associative ring with non-zero identity and all R-modules are unital. By Mod-R we denote the category of right R-modules. By a homomorphism we understand an R-homomorphism. The injective hull of an R-module A is denoted by $E(A)$.

Let Ω be a class of left R-modules and let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \qquad (1)$$

be a short exact sequence of right R-modules, where f and g are homomorphisms. If the tensor product $f \otimes_R 1_D : A \otimes_R D \to B \otimes_R D$ is a monomorphism for every $D \in \Omega$, it is said that the sequence (1) is Ω-pure. If A is a submodule of B, f is the inclusion monomorphism and the sequence (1) is Ω-pure, then A is said to be an Ω-pure submodule of B.

A right R-module M is called projective with respect to the sequence (1) if the natural homomorphism $Hom_R(M, B) \to Hom_R(M, C)$ is surjective. A right R-module is called injective with respect to the sequence (1) (or with respect to the monomorphism f) if the natural homomorphism $Hom_R(B, M) \to Hom_R(A, M)$ is

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surjective. A right R-module P is said to be Ω-pure-projective if P is projective with respect to every Ω-pure short exact sequence of right R-modules.

Following Maddox [2], a right R-module M is said to be absolutely pure if M is pure in every right R-module which contains M as a submodule.

If $\Omega = \{R/Rr \mid r \in R\}$, then an Ω-pure exact sequence (1) is called RD-pure [5].

Denote by $\mathbb{N}$ the set of natural numbers, by $\mathbb{Z}$ the ring of integers, $R^* = R \setminus \{0\}$, $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ and by $\mathcal{P}(\mathbb{N}^*)$ the set of all subsets of $\mathbb{N}^*$.

Let $\varphi : R \to \mathcal{P}(\mathbb{N}^*)$ be a function such that for every $r \in R^*$ and every $n \in \varphi(r)$, $r^n \neq 0$.

In this paper we shall consider the family of left R-modules

$$\Omega = \{R/Rr^n \mid r \in R^*, n \in \varphi(r)\}.$$

Notice that if the exact sequence (1) is RD-pure, then it is Ω-pure. Also, if $\varphi(r) = \{1\}$ for every $r \in R$, then Ω-purity is the same as RD-purity.

We shall characterize Ω-pure short exact sequences and we shall determine the structure of Ω-pure-projective right R-modules. Also, we introduce the notion of absolutely Ω-pure right R-module and we establish some properties for such modules.

2. **Basic results**

We shall recall two results which will be used later in the paper.

Theorem 2.1. [4, Proposition 2.3] *Let T be a set of right R-modules which contains a family of generators for Mod-R and let $p^{-1}(T)$ be the class of all short exact sequences in Mod-R with the property that every R-module in T is projective with respect to them. Then:*

(i) For every right R-module L there exists a short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$$

in $p^{-1}(T)$ with $M \in T$.

(ii) Every right R-module which is projective with respect to each sequence in $p^{-1}(T)$ is a direct summand of a direct sum of R-modules in T.

Lemma 2.2. [6, Lemma 7.16] *Consider the commutative diagram with exact rows in Mod-R*

$$\begin{array}{ccccccccc} & & M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & \\ 0 & \longrightarrow & N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3 & & \end{array}$$

The following statements are equivalent:

(i) There exists $\alpha : M_3 \to N_2$ *with* $g_2\alpha = \varphi_3$*;*

(ii) There exists $\beta : M_2 \to N_1$ *with* $\beta f_1 = \varphi_1$*.*

Now we can characterize Ω-pure submodules.

Theorem 2.3. *Let A be a submodule of a right R-module B. Then the following statements are equivalent:*

(i) A is Ω-pure in B.

(ii) For every $r \in R^*$ and every $n \in \varphi(r)$,

$$Ar^n = A \cap Br^n \,.$$

(iii) For every $r \in R^*$ and every $n \in \varphi(r)$, $c = br^n \in A$ for some $b \in B$ implies $c = ar^n$ for some $a \in A$.

Proof. (*i*) $\Longleftrightarrow$ (*ii*) A is Ω-pure in B if and only if for every $r \in R^*$ and every $n \in \varphi(r)$ the sequence of $\mathbb{Z}$-modules

$$0 \to A \otimes_R R/Rr^n \overset{f \otimes 1_{R/Rr^n}}{\longrightarrow} B \otimes_R R/Rr^n \overset{g \otimes 1_{R/Rr^n}}{\longrightarrow} C \otimes_R R/Rr^n \to 0 \qquad (2)$$

is exact, where $f : A \to B$ is the inclusion homomorphism. It is known the isomorphism of $\mathbb{Z}$-modules

$$D \otimes_R R/K \cong D/DK \,,$$

where D is a right R-module and K is a left ideal of R. Then the sequence (2) is exact if and only if the sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow A/Ar^n \xrightarrow{f_1} B/Br^n \xrightarrow{g_1} C/Cr^n \longrightarrow 0 \qquad (3)$$

is exact, where $f_1(a + Ar^n) = a + Br^n$ for every $a \in A$. But f_1 is injective if and only if $A \cap Br^n \subseteq Ar^n$. Since the converse inclusion is clear, it follows that A is Ω-pure in B if and only if for every $r \in R^*$ and every $n \in \varphi(r)$, we have $Ar^n = A \cap Br^n$.

$(ii) \Longrightarrow (iii)$ Assume that (ii) holds. Let $r \in R^*$ and $n \in \varphi(r)$. Then $Ar^n = A \cap Br^n$. Let $c = br^n \in A$ for some $b \in B$. Then $c \in A \cap Br^n = Ar^n$. Hence there exists $a \in A$ such that $c = ar^n$.

$(iii) \Longrightarrow (ii)$ Assume that (iii) holds. Let $r \in R^*$ and $n \in \varphi(r)$. Let $c \in A \cap Br^n$. Then there exists $a \in A$ such that $c = ar^n$. Then $c \in Ar^n$. It follows that $A \cap Br^n \subseteq Ar^n$. Therefore, $A \cap Br^n = Ar^n$. □

Theorem 2.4. *The following statements are equivalent:*

(i) The exact sequence (1) of right R-modules is Ω-pure.

(ii) For every $r \in R^*$, for every $n \in \varphi(r)$ and for every commutative diagram of right R-modules:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle k}\uparrow & & \uparrow{\scriptstyle h} \\ r^nR & \xrightarrow[v]{} & R \end{array} \qquad (4)$$

where k and h are homomorphisms and v is the inclusion homomorphism, there exists a homomorphism $w : R \to A$ such that $k = wv$.

(iii) For every $r \in R^*$ and for every $n \in \varphi(r)$, the right R-module R/r^nR is projective with respect to the exact sequence (1) of right R-modules.

Proof. We may suppose without loss of generality that A is an Ω-pure submodule of B and f is the inclusion homomorphism.

$(i) \Longrightarrow (ii)$ Assume that (i) holds. Let $r \in R^*$ and $n \in \varphi(r)$. Now consider the commutative diagram (4) of right R-modules, where v is the inclusion homomorphism. Denote $b = h(1)$ and $c = k(r^n)$. Then

$$c = fk(r^n) = hv(r^n) = h(r^n) = br^n\,.$$

By Theorem 2.3, there exists $a \in A$ such that $c = ar^n$. Define the homomorphism $w : R \to A$ by $w(1) = a$. Then

$$wv(r^n) = w(r^n) = ar^n = c = k(r^n)\,,$$

hence $k = wv$.

$(ii) \Longrightarrow (i)$ Assume that (ii) holds. Let $r \in R^*$, $n \in \varphi(r)$ and suppose that $c = br^n \in A$ for some $b \in B$. Define the homomorphisms $h : R \to B$ by $h(1) = b$ and

$k : r^nR \to A$ by $k(r^ns) = cs$ for every $s \in R$. If $r^ns = r^nt$ for some $s, t \in R$, then

$$cs - ct = c(s - t) = br^n(s - t) = 0\,,$$

hence k is well defined. Let $v : r^nR \to R$ be the inclusion homomorphism. We have

$$hv(r^n) = h(r^n) = br^n = c = fk(r^n)\,,$$

that is, $hv = fk$. Thus we obtain a commutative diagram (4). Hence there exists an homomorphism $w : R \to A$ such that $k = wv$. Denote $a = w(1)$. Then

$$c = k(r^n) = wv(r^n) = w(r^n) = ar^n\,.$$

By Theorem 2.3, the exact sequence (1) is Ω-pure.

(ii) $\Longrightarrow$ *(iii)* Assume that *(ii)* holds. Let $r \in R^*$ and $n \in \varphi(r)$. Consider the exact sequence of right R-modules

$$0 \longrightarrow r^nR \xrightarrow{v} R \xrightarrow{q} R/r^nR \longrightarrow 0 \tag{5}$$

where v is the inclusion homomorphism and q is the natural projection. Let $p : R/r^nR \to C$ be a homomorphism. Since R is projective, there exists a homomorphism $h : R \to B$ such that $gh = pq$. We have $ghv = pqv = 0$, hence there exists a homomorphism $k : r^nR \to A$ such that $hv = fk$. Hence there exists a homomorphism $w : R \to A$ such that $wv = k$. Thus we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle k} & & \uparrow{\scriptstyle h} & & \uparrow{\scriptstyle p} & & \\ 0 & \longrightarrow & r^nR & \xrightarrow[v]{} & R & \xrightarrow[q]{} & R/r^nR & \longrightarrow & 0 \end{array} \tag{6}$$

with exact rows. By Lemma 2.2, there exists a homomorphism $u : R/r^nR \to B$ such that $p = gu$. Therefore R/r^nR is projective with respect to the exact sequence (1).

(iii) $\Longrightarrow$ *(ii)* Assume that *(iii)* holds. Let $r \in R^*$ and $n \in \varphi(r)$. Consider the commutative diagram of right R-modules (4), where v is the inclusion homomorphism. We construct the exact sequence (5), where q is the natural projection. Since $ghv = gfk = 0$, there exists a homomorphism $p : R/r^nR \to C$ such that $pq = gh$. Thus we obtain a commutative diagram (6) with exact rows. Since R/r^nR is projective with respect to the sequence (1), there exists a homomorphism $u : R/r^nR \to B$ such that

$p = gu$. By Lemma 2.2, there exists a homomorphism $w : R \to A$ such that $k = wv$. □

By Theorems 2.1 and 2.4, we deduce the following two corollaries, giving the structure of Ω-pure-projective R-modules.

Corollary 2.5. *For every right R-module L there exists a short exact sequence of right R-modules*

$$0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$$

where M is Ω-pure-projective and N is an Ω-pure submodule of M.

Corollary 2.6. *Every Ω-pure-projective right R-module is a direct summand of a direct sum of R-modules of the form R/r^nR, where $r \in R$ and $n \in \varphi(r)$.*

Corollary 2.7. *Let $r \in R^*$ and $n \in \varphi(r)$. Then the following statements are equivalent:*

(i) The right ideal r^nR is Ω-pure in R.

(ii) The right ideal r^nR is a direct summand of R.

Proof. $(i) \Longrightarrow (ii)$ Assume that (i) holds. Consider the exact sequence (5) of right R-modules. By Theorem 2.4, R/r^nR is projective with respect to the sequence (5). Then the sequence (5) splits, that is, r^nR is a direct summand of R.

$(ii) \Longrightarrow (i)$ Clear. □

3. **Absolutely Ω-pure modules**

We shall give the following definition.

Definition. A right R-module A is called *absolutely Ω-pure* if A is Ω-pure in each right R-module which contains it as a submodule.

In the sequel we shall denote by $\mathcal{A}$ the class of absolutely Ω-pure right R-modules.

Theorem 3.1. *Let A be a right R-module. Then the following statements are equivalent:*

(i) $A \in \mathcal{A}$.

(ii) A is Ω-pure in $E(A)$.

(iii) For every $r \in R^*$ and $n \in \varphi(r)$, A is injective with respect to the inclusion homomorphism $v : r^n R \to R$.

Proof. $(i) \Longrightarrow (ii)$ Clear.

$(ii) \Longrightarrow (iii)$ Assume that (ii) holds. Denote $B = E(A)$ and let $r \in R^*$ and $n \in \varphi(r)$. Let $k : r^n R \to A$ be a homomorphism. Since B is injective, there exists a homomorphism $h : R \to B$ such that $hv = fk$. By Theorem 2.4, there exists a homomorphism $w : R \to A$ such that $k = wv$. Hence A is injective with respect to v.

$(iii) \Longrightarrow (i)$ Assume that (iii) holds. Let $r \in R^*$ and $n \in \varphi(r)$. Let B be a right R-module which contains A as a submodule. Consider the commutative diagram (4) of right R-modules, where f is the inclusion homomorphism. Then there exists a homomorphism $w : R \to A$ such that $wv = k$, because A is injective with respect to v. By Theorem 2.4, A is Ω-pure in B, that is, A is absolutely Ω-pure. □

Remark. Every injective right R-module is absolutely Ω-pure.

Corollary 3.2. *The class $\mathcal{A}$ is closed under taking direct products and direct summands.*

Proof. It follows as for injectivity [3, Proposition 2.2]. □

Lemma 3.3. *The class $\mathcal{A}$ is closed under taking direct sums.*

Proof. Let $(A_i)_{i \in I}$ be a family of absolutely Ω-pure right R-modules and let $A = \oplus_{i \in I} A_i$. Let $r \in R^*$ and $n \in \varphi(r)$ and let $k : r^n R \to A$ be a homomorphism. Since $k(r^n R)$ is generated by $k(r^n)$, there exists a finite subset $J \subseteq I$ such that $k(r^n R) \subseteq \oplus_{i \in J} A_i = D$. By Corollary 3.2, $D \in \mathcal{A}$. Therefore by Theorem 3.1, there exists a homomorphism $q : R \to D$ such that $qv = u$, where $u : r^n R \to D$ is the homomorphism defined by $u(r^n s) = k(r^n s)$ for every $s \in R$. Let $\alpha : D \to A$ be the inclusion homomorphism. Then $\alpha qv = \alpha u = k$. By Theorem 3.1, $A \in \mathcal{A}$. □

Theorem 3.4. *Let (1) be a short exact sequence of right R-modules and let $A, C \in \mathcal{A}$. Then $B \in \mathcal{A}$.*

Proof. Similar to the proof given for absolutely F/U-pure modules [1, Theorem 2.7]. □

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