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$\mathcal{X}$-MAXIMAL SUBGROUPS IN FINITE π-SOLVABLE GROUPS WITH RESPECT TO A SCHUNCK CLASS $\mathcal{X}$

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Abstract. Let π be an arbitrary set of primes and $\mathcal{X}$ be a π-Schunck class, i.e. $\mathcal{X}$ is a π-closed Schunck class. The paper establishes an existence and conjugacy theorem on $\mathcal{X}$-maximal subgroups in finite π-solvable groups. For the proof of the main result are used some theorems given in [4] generalizing Ore's theorems from [8]. Finally, some applications on $\mathcal{X}$-projectors in finite π-solvable groups are given.

1. **Preliminaries**

All groups considered in the paper are finite. We denote by π an arbitrary set of primes and by π' the complement to π in the set of all primes.

Some definitions will be reminded here:

Definition 1.1. A group G is *primitive* if G has a stabilizer W, i.e. a maximal subgroup W of G such that $core_G W = \{1\}$, where

$$core_G W = \cap\{W^g / g \in G\}.$$

Definition 1.2. a) A group G is π-*solvable* if any chief factor of G is either a solvable π-group or a π'-group. If π is the set of all primes, we obtain the notion of solvable group.

b) A class $\mathcal{X}$ of groups is π-*closed* if:

$$G/O_{\pi'}(G) \in \mathcal{X} \Rightarrow G \in \mathcal{X},$$

where $O_{\pi'}(G)$ denotes the largest normal π'-subgroup of G.

Definition 1.3. a) A class $\mathcal{X}$ of groups is a *homomorph* if $\mathcal{X}$ is closed under homomorphism, i.e. if $G \in \mathcal{X}$ and N is a normal subgroup of G, then $G/N \in \mathcal{X}$.

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b) A homomorph $\mathcal{X}$ is a *Schunck class* if $\mathcal{X}$ is primitively closed, i.e. if any group G, all of whose primitive factor groups are in $\mathcal{X}$, is itself in $\mathcal{X}$.

c) We shall call π-*homomorph*, respectively π-*Schunck class*, a π-closed homomorph, respectively a π-closed Schunck class.

Definition 1.4. Let $\mathcal{X}$ be a class of groups, G a group and H a subgroup of G.

a) H is an $\mathcal{X}$-*maximal subgroup* of G if: (i) $H \in \mathcal{X}$; (ii) $H \leq H^* \leq G, H^* \in \mathcal{X}$ imply $H = H^*$.

b) H is an $\mathcal{X}$-*projector* of G if for any normal subgroup N of G, HN/N is $\mathcal{X}$-maximal in G/N.

c) H is an $\mathcal{X}$-*covering subgroup* of G if: (i) $H \in \mathcal{X}$; (ii) $H \leq K \leq G$, $K_0 \lhd K$, $K/K_0 \in \mathcal{X}$ imply $K = HK_0$.

The following results will be used in the paper:

Proposition 1.5. ([1]) A solvable minimal subgroup of a finite group is abelian.

Proposition 1.6. ([6]) Let G be a group and N a subgroup of G. The following two conditions are equivalent:

(1) N is normal in G and G/N is primitive;

(2) there is a maximal subgroup W of G such that $N = core_G W$.

2. Ore's generalized theorems

In [4] are given some theorems generalizing Ore's theorems from [8]. In order to be used in the present paper, we remind them:

Theorem 2.1. *Let G be a primitive π-solvable group. If G has a minimal normal subgroup which is a solvable π-group, then G has one and only one minimal normal subgroup.*

Corollary 2.2. *If G is a primitive π-solvable group, then G has at most one minimal normal subgroup which is a solvable π-group.*

Corollary 2.3. *If a primitive π-solvable group G has a minimal normal subgroup which is a solvable π-group, then G has no minimal normal subgroups which are π'-groups.*

Theorem 2.4. *If G is a primitive π-solvable group and N is a minimal normal subgroup of G which is a solvable π-group, then $C_G(N) = N$.*

Theorem 2.5. *Let G be a π-solvable group such that:*

(i) there is a minimal subgroup M of G which is a solvable π-group and $C_G(M) = M$;

(ii) there is a minimal normal subgroup L/M of G/M such that L/M is a π'-group. Then G is primitive.

Theorem 2.6. *If G is a π-solvable group satisfying (i) and (ii) from 2.5., then any two stabilizers W_1 and W_2 of G are conjugate in G.*

Theorem 2.7. *If G is a primitive π-solvable group, $V < G$, such that there is a minimal normal subgroup M of G which is a solvable π-group and $MV = G$, then V is a stabilizer of G.*

3. An existence and conjugacy theorem on $\mathcal{X}$-maximal subgroups in finite π-solvable groups

In preparation for the main theorem we give the following lemma:

Lemma 3.1. *If G is a finite group, W is a maximal subgroup of G and $A \neq \{1\}$ is a normal subgroup of G, such that $AW = G$ and $A \cap W = \{1\}$, then A is a minimal normal subgroup of G.*

Proof. We have that $A \neq \{1\}$ is a normal subgroup of G. Let now $A^* \neq \{1\}$ be a normal subgroup of G, such that $A^* \leq A$. We shall prove that $A^* = A$. Since

$$W \leq A^*W \leq AW = G,$$

it follows that $A^*W = W$ or $A^*W = G$. But, if we suppose that $A^*W = W$, we have

$$A^* \subseteq A \cap W = \{1\},$$

hence the contradiction $A^* = \{1\}$. So $A^*W = G$. In order to prove that $A^* = A$, suppose that $A^* < A$. This means that there is an element $a \in A \setminus A^* \subseteq G = A^*W$. Then $a = a^*w$, with $a^* \in A^*$, $w \in W$. It follows that

$$w = (a^*)^{-1}a \in A \cap W = \{1\},$$

hence $w = 1$, which implies the contradiction $a = a^* \in A^*$. So $A^* = A$. $\square$

The main theorem of this paper is the following:

Theorem 3.2. *Let $\mathcal{X}$ be a π-Schunck class, G a π-solvable group and A an abelian normal subgroup of G with $G/A \in \mathcal{X}$. Then:*

(1) there is a subgroup S of G with $S \in \mathcal{X}$ and $AS = G$;

(2) there is an $\mathcal{X}$-maximal subgroup S of G with $AS = G$;

(3) if S_1 and S_2 are $\mathcal{X}$-maximal subgroups of G with $AS_1 = G = AS_2$, then S_1 and S_2 are conjugate in G.

Proof.

(1) Let

$$\mathcal{S} = \{S^*/S^* \leq G, AS^* = G\}.$$

Since $G \in \mathcal{S}$, we have $\mathcal{S} \neq \emptyset$. Considering $\mathcal{S}$ ordered by inclusion and applying Zorn's lemma, $\mathcal{S}$ has a minimal element S. Obviously, $AS = G$.

We shall prove that $S \in \mathcal{X}$.

Put $D = S \cap A$. Let us notice that D is a normal subgroup of G. Indeed, if $g \in G$ and $d \in D$, we have $g = as$, with $a \in A$, $s \in S$, and so, A being abelian and D being normal in S,

$$g^{-1}dg = (as)^{-1}d(as) = s^{-1}a^{-1}das = s^{-1}a^{-1}ads = s^{-1}ds \in D.$$

Let W be a maximal subgroup of S. Then $D \leq W$, else $DW \neq W$, hence

$$W < DW \leq S$$

and so $DW = S$. But this implies

$$G = AS = ADW = AW,$$

which means that $W \in \mathcal{S}$, in contradiction with the minimality of S in $\mathcal{S}$.

Put $N = core_S W$. We have $D \leq N$. Indeed, from $D \leq W$ it follows that $D = D^g \leq W^g$ for any $g \in S$, hence $D \leq core_S W = N$. Then

$$S/N \cong (S/D)/(N/D).$$

But

$$S/D = S/S \cap A \cong AS/A = G/A \in \mathcal{X}.$$

$\mathcal{X}$ being a homomorph, it follows that $S/N \in \mathcal{X}$.

For any primitive factor group S/N of S, we have $S/N \in \mathcal{X}$. Indeed, S/N being primitive, it follows from 1.6. that there is a maximal subgroup W of S such that $N = core_S W$. But we proved that in this case we have $S/N \in \mathcal{X}$. This means that any primitive factor group S/N of S is in $\mathcal{X}$. The primitive closure of $\mathcal{X}$ leads now to $S \in \mathcal{X}$. Thus (1) is proved.

(2) Let now

$$\mathcal{S}^* = \{S/S \leq G, S \in \mathcal{X}, AS = G\}$$

ordered by inclusion. Because of (1), $\mathcal{S}^* \neq \emptyset$. By Zorn's lemma, $\mathcal{S}^*$ has a maximal element $S \in \mathcal{S}^*$. Obviously, $S \leq G$, $S \in \mathcal{X}$, $AS = G$. We shall prove that S in an $\mathcal{X}$-maximal subgroup of G. Let $S \leq S^* \leq G$, with $S^* \in \mathcal{X}$. Then $S = S^*$, as the following considerations show: from $AS = G$ it follows that $AS^* = G$ and so $S^* \in \mathcal{S}^*$; but $S \leq S^*$, $S^* \in \mathcal{S}^*$ imply by the maximality of S that $S = S^*$.

(3) Let S_1 and S_2 be $\mathcal{X}$-maximal subgroups of G with $AS_1 = G = AS_2$. We shall prove by induction on $|G|$ that S_1 and S_2 are conjugate in G.

Let us distinguish two cases:

a) $G \in \mathcal{X}$. S_1 and S_2 being $\mathcal{X}$-maximal subgroups of G, we have $S_1 = G = S_2$ and so S_1 and S_2 are conjugate in G.

b) $G \notin \mathcal{X}$. It means that there is a primitive factor group G/N with $G/N \notin \mathcal{X}$, else the primitive closure of $\mathcal{X}$ leads to the contradiction $G \in \mathcal{X}$. We also have $NS_1 \neq G$ and $NS_2 \neq G$. Indeed, if we suppose, for example, that $NS_1 = G$, we obtain

$$NS_1/N = G/N \notin \mathcal{X}$$

and on the other side

$$NS_1/N \cong S_1/S_1 \cap N \in \mathcal{X}.$$

Let us prove that AN/N is minimal normal subgroup of G/N. The factor group G/N being primitive, we apply 1.6. and there is a minimal subgroup W of G with $N = core_G W$. We have $A \not\subseteq W$, because supposing that $A \leq W$ it follows that for any $g \in G$, $A = A^g \leq W^g$, hence

$$A \leq \cap\{W^g/g \in G\} = core_G W = N$$

and

$$G/A = AS_1/A \cong S_1/A \cap S_1 \in \mathcal{X}$$

and so

$$G/N \cong (G/A)/(N/A) \in \mathcal{X},$$

in contradiction with $G/N \notin \mathcal{X}$. Put $A_1 = A \cap W$. Since W is a maximal subgroup of G, we have $AW = W$ or $AW = G$. But $AW = W$ implies $A \leq W$, a contradiction. So $AW = G$. It is easy to prove that A_1 is a normal subgroup of G. Indeed, if $g \in G = AW$ and $a_1 \in A_1$, put $g = aw$, with $a \in A$, $w \in W$ and, A being abelian and A_1 being normal in W, we have:

$$g^{-1}a_1g = (aw)^{-1}a_1(aw) = w^{-1}a^{-1}a_1aw = w^{-1}a_1a^{-1}aw = w^{-1}a_1w \in A_1.$$

We are now in the hypotheses of lemma 3.1. Indeed, W/A_1 is a maximal subgroup of G/A_1, A/A_1 is a normal $\neq \{1\}$ subgroup of G/A_1 satisfying:

$$A/A_1 \cdot W/A_1 = G/A_1 \text{ and } A/A_1 \cap W/A_1 = \{1\}.$$

It follows that A/A_1 is a minimal normal subgroup of G/A_1. From this and from the isomorphism

$$AN/N \cong A/A_1$$

we obtain that AN/N is minimal normal subgroup of G/N.

Denote by $M = AN$. It follows that for $i = 1, 2$, we have

$$(NS_i)M = (NS_i)(AN) = G.$$

Furthermore, NS_i/N is a stabilizer of G/N, for $i = 1, 2$. In order to prove this, we use theorem 2.7. In the primitive π-solvable group G/N, we consider $NS_i/N < G/N$ and $M/N = AN/N$ minimal normal subgroup of G/N. Obviously, $M/N \cdot NS_i/N = G/N$. It remains to prove that M/N is a solvable π-group. Being a minimal normal subgroup of the π-solvable group G/N, M/N is either a solvable π-group or a π'-group. If we suppose that M/N is a π'-group, we obtain

$$M/N \leq O_{\pi'}(G/N) \leq G/N$$

and

$$(G/N)/O_{\pi'}(G/N) \cong ((G/N)/(M/N))/(O_{\pi'}(G/N)/(M/N)).$$

But $\mathcal{X}$ being a homomorph, we have

$$(G/N)/(M/N) \cong G/M \cong G/AN \cong (G/A)/(AN/A) \in \mathcal{X}$$

and

$$(G/N)/O_{\pi'}(G/N) \in \mathcal{X},$$

hence by the π-closure of the class $\mathcal{X}$ we obtain the contradiction $G/N \in \mathcal{X}$. It follows that M/N is a solvable π-group. Applying 2.7., NS_i/N is a stabilizer of G/N.

The next step in our proof is to show that NS_1/N and NS_2/N are conjugate in G/N. For this, we apply theorem 2.6. to the π-solvable group G/N. Indeed, G/N satisfies the conditions (i) and (ii) from theorem 2.5., as we prove below:

(i) $M/N = AN/N$ is minimal normal subgroup of G/N, such that M/N is a solvable π-group and $C_{G/N}(M/N) = M/N$. The last condition follows from theorem 2.4. applied to the primitive π-solvable group G/N and its minimal normal subgroup M/N which is a solvable π-group.

(ii) There is a minimal normal subgroup $(L/N)/(M/N)$ of $(G/N)/(M/N)$, such that $(L/N)/(M/N)$ is a π'-group. Indeed, if we suppose the contrary, then any minimal normal subgroup $(L/N)/(M/N)$ of the π-solvable group $(G/N)/(M/N)$ is a solvable π-group. But M/N being a solvable π-group, it follows that L/N is also a solvable π-group. Theorem 2.1. applied to the primitive π-solvable group G/N, which has the minimal normal subgroup M/N such that M/N is a solvable π-group, leads to the conclusion that G/N has one and only one minimal subgroup. Since L/N is $a \neq \{1\}$ normal subgroup of G/N, two possibilities can happen:

1) L/N is a minimal normal subgroup of G/N. It follows that $M/N = L/N$, in contradiction with the assumption that $(L/N)/(M/N)$ is a minimal normal subgroup of $(G/N)/(M/N)$.

2) L/N is not a minimal normal subgroup of G/N and so $M/N < L/N$. But this also leads to a contradiction, as the following shows:

$$G/N = M/N \cdot NS_1/N < L/N \cdot NS_1/N = G/N.$$

We are now in the hypotheses of theorem 2.6., hence NS_1/N and NS_2/N are conjugate in G/N. It follows that

$$NS_1 = (NS_2)^g = NS_2^g,$$

where $g \in G$.

Denote by

$$G^* = NS_1 = NS_2^g$$

and by

$$A^* = A \cap G^*.$$

We can now apply the induction for G^*, where $G^* = NS_1 < G$. Indeed, A^* is an abelian normal subgroup of G^*, with

$$G^*/A^* = G^*/A \cap G^* \cong AG^*/A = ANS_1/A = G/A \in \mathcal{X}$$

and S_1 and S_2^g are $\mathcal{X}$-maximal subgroups in G^*. We also have:

$$A^*S_1 = (A \cap G^*)S_1 = S_1(A \cap G^*) = (S_1A) \cap G^* = G \cap G^* = G^*$$

and

$$A^*S_2^g = (A \cap G^*)S_2^g = S_2^g(A \cap G^*) = (S_2^gA) \cap G^* = (S_2A)^g \cap G^* = G \cap G^* = G^*.$$

By the induction, S_1 and S_2^g are conjugate in G^*. It follows that S_1 and S_2 are conjugate in G. □

Remarks. a) Theorem 3.2. was earlier establishes in [2], but the proof was based on some of R. Baer's theorems from [1]. In the present paper, we give a new proof, based on Ore's generalized theorems given in [4].

b) Particularly, for π the set of all primes, we obtain from theorem 3.2. a theorem given in [6] by W. Gaschütz.

4. **Projectors in finite π-solvable groups**

Theorem 3.2 is important for the study of projectors in finite π-solvable groups, as the following result (given in [3]) shows:

Theorem 4.1. *If $\mathcal{X}$ is a π-Schunck class, then any two $\mathcal{X}$-projectors of a π-solvable group G are conjugate in G.*

Proof. By induction on $|G|$. We remind the proof from [3]:

Let S_1 and S_2 be two $\mathcal{X}$-projectors of G. Let M be a minimal normal subgroup of G. Put $S_1^* = MS_1$ and $S_2^* = MS_2$. Applying the induction for G/M, we obtain that S_1^*/M and S_2^*/M are conjugate in G/M, hence S_1^* and S_2^* are conjugate in G, i.e. $S_1^* = (S_2^*)^g$, with $g \in G$.

We prove that S_1 and S_2 are conjugate in G, considering the two cases which are possible for the minimal normal subgroup M of the π-solvable group G:

1) M is a solvable π-group. Then by 1.5., M is abelian. We are now in the hypotheses of theorem 3.2.: S_1^* is π-solvable, where

$$S_1^* = MS_1 = MS_2^g,$$

M is a normal abelian subgroup of S_1^*, with $S_1^*/M \in \mathcal{X}$ and S_1 and S_2^g are $\mathcal{X}$-maximal subgroups in S_1^*. Applying theorem 3.2., we deduce that S_1 and S_2^g are conjugate in S_1^*. It follows that S_1 and S_2 are conjugate in G.

2) M is a π'-group. Then

$$M \leq O_{\pi'}(S_1^*)$$

and

$$S_1^*/O_{\pi'}(S_1^*) \cong (S_1^*/M)/(O_{\pi'}(S_1^*)/M) \in \mathcal{X},$$

which imply by the π-closure of $\mathcal{X}$ that $S_1^* \in \mathcal{X}$. Hence by the fact that S_1 and S_2^g are $\mathcal{X}$-maximal in S_1^*, we obtain $S_1 = S_1^* = S_2^g$. $\square$

The conjugacy theorem 4.1. on projectors can be completed with an existence theorem. In [5], we proved by means of Ore's generalized theorems ([4]) the following result:

Lemma 4.2. *Let $\mathcal{X}$ be a π-homomorph. $\mathcal{X}$ is a Schunck class is and only if any finite π-solvable group G has $\mathcal{X}$-covering subgroups.*

It is well-known that for a homomorph $\mathcal{X}$ and a finite group G, any $\mathcal{X}$-covering subgroup of G is also an $\mathcal{X}$-projector of G. Thus lemma 4.2. leads to the following existence theorem on projectors:

Theorem 4.3. *If $\mathcal{X}$ is a π-Schunck class, then any finite π-solvable group G has $\mathcal{X}$-projectors.*

In [3] we proved the following result:

Lemma 4.4. *A π-homomorph $\mathcal{X}$ with the property that any finite π-solvable group has $\mathcal{X}$-projectors is a Schunck class.*

Theorem 4.5. *Let $\mathcal{X}$ be a π-homomorph. $\mathcal{X}$ is a Schunck class if and only if any finite π-solvable group has $\mathcal{X}$-projectors.*

References

[1] Baer, R., *Classes of finite groups and their properties,* Illinois J. Math., **1**, No. 4, 1957, 115-187.
[2] Covaci, R., *Some properties of projectors in finite π-solvable groups,* Studia Univ. "Babeş-Bolyai", Math., **XXVI**, No. 1, 1981, 5-8
[3] Covaci, R., *Projectors and covering subgroups,* Studia Univ. "Babeş-Bolyai", Math., **XXVII**, 1982, 33-36.
[4] Covaci, R., *A generalization of some of Ore's theorems,* Studia Univ. "Babeş-Bolyai", Math., **XLIII**, No. 1, 1998, 15-21.
[5] Covaci, R., *Some applications of Ore's generalized theorems in the formation theory,* Studia Univ. "Babeş-Bolyai", Math., **XLIV**, No. 4, 1999, 3-9.
[6] Gaschütz, W., *Selected topics in the theory of soluble groups,* Australian National University, Canberra, Jan.-Feb. 1969.
[7] Huppert, B., *Endliche Gruppen I,* Springer Verlag, Berlin-New York, 1967.
[8] Ore, O., *Contributions to the theory of groups of finite order,* Duke Math. J., 5, 1939, 431-450.

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