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ON NEARLY-COSYMPLECTIC HYPERSURFACES IN NEARLY-KÄHLERIAN MANIFOLDS

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Abstract. It is proved that the type number of a nearly-cosymplectic hypersurface in a nearly-Kählerian manifold is at most one. It is also proved that such a hypersurface is minimal if and only if it is totally geodesic.

1. **Introduction**

The theory of almost contact metric structures occupies one of the leading places in modern differential-geometrical researches. It is due to a number of its applications in modern mathematical physics (e.g. in classical mechanics [1] and in theory of geometrical quantization [10]) and to the riches of the internal contents of the theory as well, and also to its close connection with other sections of geometry.

One of the most important examples of almost contact metric structures, wich appreciably determines their role in differential geometry, is the structure induced on an oriented hypersurface in an almost Hermitian manifold. Well known scientists such as D.E. Blair, S. Goldberg, V.F. Kirichenko, S. Sasaki, S. Tanno were engaged in studying almost contact metric hypersurfaces in almost Hermitian manifolds.

In the present note, nearly-cosymplectic hypersurfaces in nearly-Kählerian manifolds are considered. We can mention that the class of nearly-Kählerian manifolds is one of the most important classes of almost Hermitian manifolds [9]. A great number of significant works is devoted to its studying. For not going in details of such an extensive subject, we remark only, that the six-dimensional sphere with a nearly-Kählerian structure is considered in [7], [8], [11], [15] etc.

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The present work is a continuation of researches of the author, who studied cosymplectic hypersurfaces in six-dimensional submanifolds of Cayley algebra before (see [3], [4]).

2. **Preliminaries**

We consider an almost Hermitian (AH) manifold, i.e. a $2n$-dimensional manifold M^{2n} with a Riemannian metric $g = \langle \cdot, \cdot \rangle$ and an almost complex structure J. Moreover, the following condition must hold

$$\langle JX, JY \rangle = \langle X, Y \rangle, \quad X, Y \in \aleph(M^{2n}),$$

where $\aleph(M^{2n})$ is the module of smooth vector fields on M^{2n}. All considered manifolds, tensor fields and similar objects are assumed to be of the class C^{∞}. We recall that the fundamental (or Kählerian [14]) form of an almost Hermitian manifold is determined by

$$F(X, Y) = \langle X, JY \rangle, \quad X, Y \in \aleph(M^{2n}).$$

Let $\left(M^{2n}, J, g = \langle \cdot, \cdot \rangle\right)$ be an arbitrary almost Hermitian manifold. We fix a point $p \in M^{2n}$. As $T_p(M^{2n})$ we denote the tangent space at the point p, $\{J_p, g_p = \langle \cdot, \cdot \rangle\}$ is the almost Hermitian structure at the point p induced by the structure $\{J, g = \langle \cdot, \cdot \rangle\}$. The frames adapted to the structure (or A-frames) look as follows [2]

$$(p, \varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{\hat{1}}, \ldots, \varepsilon_{\hat{n}}),$$

where ε_a are the eigenvectors corresponded to the eigenvalue $i = \sqrt{-1}$, and $\varepsilon_{\hat{a}}$ are the eigenvectors corresponded to the eigenvalue $-i$, $\varepsilon_{\hat{a}} = \overline{\varepsilon_a}$. Here the indice a ranges from 1 to n, and we state $\hat{a} = a + n$.

The matrix of the operator of the almost complex structure written in an A-frame looks as follows:

$$\left(J_j^k\right) = \left(\begin{array}{c|c} iI_n & 0 \\ \hline 0 & -iI_n \end{array}\right),$$

where I_n is the identity matrix; $k, j = 1, \ldots, 2n$. By direct computing, it is easy to obtain that the matrixes of the metric g and of the fundamental form F in an A-frame look as follows, respectively:

$$(g_{kj}) = \left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right), \quad (F_{kj}) = \left(\begin{array}{c|c} 0 & iI_n \\ \hline -iI_n & 0 \end{array}\right).$$

As it is well-known [9], an almost Hermitian manifold is called nearly-Kählerian (NK), if

$$\nabla_X(J)Y + \nabla_Y(J)X = 0, \quad X, Y \in \aleph(M^{2n}),$$

where ∇ is the Levi-Civita connection of the metric.

Let N be an oriented hypersurface in an almost Hermitian manifold M^{2n}, and let σ be the second fundamental form of the immersion of N into M^{2n}. As it is well-known [16], the almost Hermitian structure on M^{2n} induces an almost contact metric structure on N. We recall [16], that an almost contact metric structure on an odd-dimensional manifold N is defined by the system $\{\Phi, \xi, \eta, g\}$ of tensor fields on this manifold, where ξ is a vector, η is a covector, Φ is a tensor of the type $(1,1)$ and g is a Riemannian metric on N such that

$$\eta(\xi) = 1, \quad \Phi(\xi) = 0, \quad \eta \circ \Phi = 0, \quad \Phi^2 = -id + \xi \otimes \eta,$$

$$\langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y), \quad X, Y \in \aleph(N).$$

The almost contact metric structure is called nearly-cosymplectic, if

$$\nabla_X(\Phi)Y + \nabla_Y(\Phi)X = 0, \quad \nabla_X(\eta)Y + \nabla_Y(\eta)X = 0, \quad X, Y \in \aleph(N).$$

At the end of this section, note that when we give a Riemannian manifold and its submanifold, the rank of the determined second fundamental form is called the type number (see, for example, [12]).

3. **Three theorems**

Now, we can state the main results of this work.

THEOREM A. *The type number of a nearly-cosymplectic hypersurface in a nearly-Kählerian manifold is at most one.*

Proof.

Let N be an oriented hypersurface in a nearly-Kählerian manifold M^{2n}. We use the first group of Cartan structural equations of an almost contact metric structure induced on a hypersurface in an almost Hermitian manifold [16]:

$$
\begin{gathered}
d\omega^\alpha = \omega^\alpha_\beta \wedge \omega^\beta + B^{\alpha\beta}{}_\gamma \omega^\gamma \wedge \omega_\beta + B^{\alpha\beta\gamma}\omega_\beta \wedge \omega_\gamma + \left(\sqrt{2}B^{\alpha n}{}_\beta + i\sigma^\alpha_\beta\right)\omega^\beta \wedge \omega + \\
+\left(-\sqrt{2}\widetilde{B}^{n\alpha\beta} - \frac{1}{\sqrt{2}}B^{\alpha\beta n} - \frac{1}{\sqrt{2}}B^{\alpha\beta}{}_n + i\sigma^{\alpha\beta}\right)\omega_\beta \wedge \omega, \\
d\omega_\alpha = -\omega^\beta_\alpha \wedge \omega_\beta + B_{\alpha\beta}{}^\gamma \omega_\gamma \wedge \omega^\beta + B_{\alpha\beta\gamma}\omega^\beta \wedge \omega^\gamma + \left(\sqrt{2}B_{\alpha n}{}^\beta - i\sigma^\beta_\alpha\right)\omega_\beta \wedge \omega + \\
+\left(-\sqrt{2}\widetilde{B}_{n\alpha\beta} - \frac{1}{\sqrt{2}}B_{\alpha\beta n} - \frac{1}{\sqrt{2}}B_{\alpha\beta}{}^n - i\sigma_{\alpha\beta}\right)\omega^\beta \wedge \omega, \qquad (1) \\
d\omega = \sqrt{2}B_{n\alpha\beta}\omega^\alpha \wedge \omega^\beta + \sqrt{2}B^{n\alpha\beta}\omega_\alpha \wedge \omega_\beta + \\
+\left(\sqrt{2}B^{n\alpha}{}_\beta - \sqrt{2}B_{n\beta}{}^\alpha - 2i\sigma^\alpha_\beta\right)\omega^\beta \wedge \omega_\alpha + \\
+\left(\widetilde{B}_{n\beta n} + B_{n\beta}{}^n + i\sigma_{n\beta}\right)\omega \wedge \omega^\beta + \left(\widetilde{B}^{n\beta n} + B^{n\beta}{}_n - i\sigma^\beta_n\right)\omega \wedge \omega_\beta,
\end{gathered}
$$

where

$$
\begin{gathered}
\widetilde{B}^{abc} = -\frac{i}{2}J^a_{\hat{b},\hat{c}}, \quad \widetilde{B}_{abc} = \frac{i}{2}J^{\hat{a}}_{b,c}; \\
B^{abc} = -\widetilde{B}^{a[bc]}, \quad B_{abc} = -\widetilde{B}_{a[bc]}; \\
B^{ab}{}_c = -\frac{i}{2}J^a_{\hat{b},c}, \quad B_{ab}{}^c = \frac{i}{2}J^{\hat{a}}_{b,\hat{c}}.
\end{gathered}
$$

Here and further, the indices a, b, c range from 1 to n and the indices α, β, γ range from 1 to $n-1$; $\hat{a} = a + n$. $\left\{B^{abc}\right\}$, $\{B_{abc}\}$ and $\left\{B^{ab}{}_c\right\}$, $\{B_{ab}{}^c\}$ are the components of Kirichenko virtual (KV) and Kirichenko structural (KS) tensors, respectively [5].

Taking into account that an almost Hermitian structure is nearly-Kählerian if and only if [6]

$$B^{abc} + B^{acb} = 0, \quad B_{abc} + B_{acb} = 0, \quad B^{ab}{}_c = 0, \quad B_{ab}{}^c = 0,$$

we can rewrite the Cartan structural equations (1) in the following form:

$$
\begin{gathered}
d\omega^\alpha = \omega^\alpha_\beta \wedge \omega^\beta + B^{\alpha\beta\gamma}\omega_\beta \wedge \omega_\gamma + i\sigma^\alpha_\beta \omega^\beta \wedge \omega + \\
+\left(-\sqrt{2}\widetilde{B}^{n\alpha\beta} - \frac{1}{\sqrt{2}}B^{\alpha\beta n} + i\sigma^{\alpha\beta}\right)\omega_\beta \wedge \omega, \\
d\omega_\alpha = -\omega^\beta_\alpha \wedge \omega_\beta + B_{\alpha\beta\gamma}\omega^\beta \wedge \omega^\gamma - i\sigma^\beta_\alpha \omega_\beta \wedge \omega + \\
+\left(-\sqrt{2}\widetilde{B}_{n\alpha\beta} - \frac{1}{\sqrt{2}}B_{\alpha\beta n} - i\sigma_{\alpha\beta}\right)\omega^\beta \wedge \omega, \qquad (2) \\
d\omega = \sqrt{2}B_{n\alpha\beta}\omega^\alpha \wedge \omega^\beta + \sqrt{2}B^{n\alpha\beta}\omega_\alpha \wedge \omega_\beta -
\end{gathered}
$$

$$-2i\sigma^{\alpha}_{\beta}\omega^{\beta}\wedge\omega_{\alpha}+\left(\widetilde{B}_{n\beta n}+i\sigma_{n\beta}\right)\omega\wedge\omega^{\beta}+\left(\widetilde{B}^{n\beta n}-i\sigma^{\beta}_{n}\right)\omega\wedge\omega_{\beta}.$$

Compareing (2) with the Cartan structural equations of a nearly-cosymplectic structure [16]:

$$\begin{aligned} d\omega^{\alpha} &= \omega^{\alpha}_{\beta}\wedge\omega^{\beta}+D^{\alpha\beta\gamma}\omega_{\beta}\wedge\omega_{\gamma}+D^{\alpha\beta}\omega_{\beta}\wedge\omega,\\ d\omega_{\alpha} &= -\omega^{\beta}_{\alpha}\wedge\omega_{\beta}+D_{\alpha\beta\gamma}\omega^{\beta}\wedge\omega^{\gamma}+D_{\alpha\beta}\omega^{\beta}\wedge\omega,\\ d\omega &= -\frac{2}{3}D_{\alpha\beta}\omega^{\alpha}\wedge\omega^{\beta}-\frac{2}{3}D^{\alpha\beta}\omega_{\alpha}\wedge\omega_{\beta}, \end{aligned} \tag{3}$$

where

$$D^{\alpha\beta\gamma}=\frac{i}{2}\Phi^{\alpha}_{[\widehat{\beta},\widehat{\gamma}]},\quad D_{\alpha\beta\gamma}=-\frac{i}{2}\Phi^{\widehat{\alpha}}_{[\beta,\gamma]},$$
$$D^{\alpha\beta}=\frac{3}{2}i\Phi^{\alpha}_{\widehat{\beta},n},\quad D_{\alpha\beta}=-\frac{3}{2}i\Phi^{\widehat{\alpha}}_{\beta,n},$$

we get the conditions, whose simultaneous fulfilment is a criterion for the structure on N to be nearly-cosymplectic:

$$1)\ B^{\alpha\beta\gamma}=D^{\alpha\beta\gamma},\quad 2)\ -\frac{3}{\sqrt{2}}\widetilde{B}^{n\alpha\beta}+i\sigma^{\alpha\beta}=-D^{\alpha\beta},\quad 3)\ \sqrt{2}B^{n\alpha\beta}=-\frac{2}{3}D^{\alpha\beta},$$
$$4)\ \sigma^{\alpha}_{\beta}=0,\quad 5)\ \sigma^{\beta}_{n}=0 \tag{4}$$

and the formulae, obtained by complex conjugation (no need to write them down explicitly).

From $(4)_3$ we have

$$D^{\alpha\beta}=-\frac{3}{\sqrt{2}}B^{n\alpha\beta}.$$

We substitute this value in $(4)_2$:

$$-\frac{3}{\sqrt{2}}\widetilde{B}^{n\alpha\beta}+i\sigma^{\alpha\beta}=\frac{3}{\sqrt{2}}B^{n\alpha\beta}.$$

Since

$$B^{n\alpha\beta}=-\widetilde{B}^{n[\alpha\beta]}=-\frac{1}{2}\left(\widetilde{B}^{n\alpha\beta}-\widetilde{B}^{n\beta\alpha}\right)=-\widetilde{B}^{n\alpha\beta},$$

we obtain $\sigma_{\alpha\beta}=0$. That is why we can rewrite the conditions (4) as follows:

$$1)\ B^{\alpha\beta\gamma}=D^{\alpha\beta\gamma},\ \ 2)\ B^{n\alpha\beta}=-\frac{\sqrt{2}}{3}D^{\alpha\beta},\ \ 3)\ \sigma^{\alpha\beta}=0,$$
$$4)\ \sigma^{\alpha}_{\beta}=0,\ \ 5)\ \sigma^{\beta}_{n}=0. \tag{5}$$

We have that the conditions

$$\sigma^{\alpha\beta}=\sigma^{\alpha}_{\beta}=\sigma^{\beta}_{n}=0$$

are necessary for the structure, induced on an oriented hypersurface in a nearly-Kählerian manifold M^{2n}, to be nearly-cosymplectic. So, the matrix of the second fundamental form of the immersion of a nearly-cosymplectic hypersurface into a nearly-Kählerian manifold looks as follows:

$$(\sigma_{ps}) = \left(\begin{array}{c|c|c} & 0 & \\ 0 & \vdots & 0 \\ & 0 & \\ \hline 0\dots0 & \sigma_{nn} & 0\dots0 \\ \hline & 0 & \\ 0 & \vdots & 0 \\ & 0 & \end{array}\right), \quad p,s = 1,\dots,2n-1. \tag{6}$$

As it is evident, *rank* $\sigma \leq 1$, i.e. the type number of the hypersurface is at most one, Q.E.D.

$\square$

Considering the matrix of the second fundamental form of the immersion of N into M^{2n}, we come to another result.

THEOREM B. *A nearly cosymplectic hypersurface N in a nearly-Kählerian manifold M^{2n} is minimal if and only if*

$$\sigma(\xi,\xi) = 0.$$

Proof.

Let us use a criterion of minimality for an arbitrary hypersurface [13]

$$g^{ps}\sigma_{ps} = 0, \quad p,s = 1,\dots,2n-1.$$

Knowing how the matrix of the contravariant metric tensor looks [16]

$$(g^{ps}) = \left(\begin{array}{c|c|c} 0 & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & I_\alpha \\ \hline 0\dots0 & 1 & 0\dots0 \\ \hline I_\alpha & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & 0 \end{array}\right),$$

we have

$$g^{ps}\sigma_{ps} = g^{\alpha\beta}\sigma_{\alpha\beta} + g^{\widehat{\alpha}\widehat{\beta}}\sigma_{\widehat{\alpha}\widehat{\beta}} + g^{\widehat{\alpha}\beta}\sigma_{\widehat{\alpha}\beta} + g^{\alpha\widehat{\beta}}\sigma_{\alpha\widehat{\beta}} + g^{\alpha n}\sigma_{\alpha n} + g^{\widehat{\alpha}n}\sigma_{\widehat{\alpha}n} + g^{nn}\sigma_{nn} = \sigma_{nn}.$$

Therefore $g^{ps}\sigma_{ps} = 0 \Leftrightarrow \sigma_{nn} = 0$. The equality $\sigma_{nn} = 0$ means that $\sigma(\xi,\xi) = 0$, Q.E.D.

$\sqcup$

THEOREM C. *If N is a nearly-cosymplectic hypersurface in a nearly-Kählerian manifold M^{2n} and t is its type number, then the following statements are equivalent:*

1) N is a minimal hypersurface in M^{2n},

2) N is a totally geodesic hypersurface in M^{2n},

3) $t \equiv 0$.

Proof.

If a nearly-cosymplectic hypersurface is minimal, then in view of THEOREM B $\sigma_{nn} = \sigma(\xi,\xi) = 0$, and consequently the matrix (6) vanishes. This indicates that the hypersurface is totally geodesic. It is evident that type number vanishes at its every point.

Conversely, if $t \equiv 0$, then the matrix of the second fundamental form vanishes, i.e. the hypersurface is totally geodesic. As $\sigma_{nn} = 0$, N is a minimal hypersurface in M^{2n}.

$\sqcup$

4. **Some additional results**

Taking into account that the class of nearly-Kählerian manifolds contains all Kählerian manifolds [9] as well as the class of nearly-cosymplectic manifolds contains all cosymplectic manifolds [16], by force of THEOREM A we come to the following results:

Corollary 1A. The type number of a nearly-cosymplectic hypersurface in a Kählerian manifold is at most one.

Corollary 2A. The type number of a cosymplectic hypersurface in a nearly-Kählerian manifold is at most one.

Corollary 3A. The type number of a cosymplectic hypersurface in a Kählerian manifold is at most one.

Similarly, by force of THEOREM B and THEOREM C we have:

Corollary 1B (2B, 3B). A nearly-cosymplectic (cosymplectic, cosymplectic) hypersurface in a Kählerian (nearly-Kählerian, Kählerian) manifold is minimal if and only if $\sigma(\xi, \xi) = 0$.

Corollary 1C (2C, 3C). If N is a nearly-cosymplectic (cosymplectic, cosymplectic) hypersurface in a Kählerian (nearly-Kählerian, Kählerian) manifold M^{2n} and t is its type number, then the following statements are equivalent:

1) N is a minimal hypersurface in M^{2n},

2) N is a totally geodesic hypersurface in M^{2n},

3) $t \equiv 0$.

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