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FUNCTIONALS WHICH SATISFY A MAXIMUM PRINCIPLE

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Abstract. The purpose of this paper is to present some examples of functionals, defined on the solutions of an elliptic equation, which satisfy a maximum principle.

1. **Introduction**

Let Ω be a domain in $\mathbb{R}^n$ with boundary $\partial\Omega$. Let us consider the following differential operator:

$$Lu := \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i,j=1}^{n} b_i \frac{\partial}{\partial x_i} + c \tag{1}$$

We assume that L satisfies the following maximum principles ([1]):

MP: There is a subset $\Gamma \subset \partial\Omega$ such that, if:

1. $u \in C(\overline{\Omega})$
2. the derivatives of u occurring in L are continuous in $\overline{\Omega}\backslash\Gamma$
3. $Lu \geq 0$,in $\overline{\Omega}\backslash\Gamma$

then $\sup\limits_{\overline{\Omega}} \varphi(u) = \sup\limits_{\Gamma} \varphi(u)$

Let us consider the following system:

$$Lu_k + f_k(x,u) = 0,\ k = \overline{1,m},\ x \in \Omega. \tag{2}$$

Let $\varphi \in C^2(\mathbb{R}^m)$. The following result is given in [3] (see also [1]):

Theorem 1.1. *Let u be a solution of (2). If:*

(i) the hessian of φ is positive semidefinite,

(ii) $-\sum\limits_{k=1}^{m} \frac{\partial\varphi(y)}{\partial y_k} f_k(x,y) + c(x)\left[\varphi(y) - \sum\limits_{k=1}^{m} \frac{\partial\varphi(y)}{y_k} y_k\right] \geq 0, \forall y \in \mathbb{R}^m,$

then $\sup\limits_{\overline{\Omega}} \varphi(u) = \sup\limits_{\Gamma} \varphi(u)$

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The purpose of this paper is to use the Theorem 1.1 for constructing functionals defined on the solution of system (2), which satisfy a maximum principle.

If $c = 0$, then the condition (ii) from Theorem 1.1, becomes:

$$-\sum_{k=1}^{m} \frac{\partial\varphi(y)}{y_k} f_k(x,y) \geq 0, \forall y \in \mathbb{R}^m$$

$$\sum_{k=1}^{m} \frac{\partial\varphi(y)}{y_k} f_k(x,y) \leq 0, \forall y \in \mathbb{R}^m$$

$$\frac{\partial\varphi}{\partial y_1} f_1 + ... + \frac{\partial\varphi}{\partial y_m} f_m \leq 0, \forall y \in \mathbb{R}^m$$

We assume $f_k(x,y) = f_k(y)$, and we can choose φ by solving the partial differential equation:

$$\frac{dy_1}{f_1} = \frac{dy_2}{f_2} = ... = \frac{dy_m}{f_m} \tag{3}$$

in the form $\varphi(y) = k$, where k is a constant.

2. **Examples of functionals which satisfies MP**

We will consider the system given in [1]

$$\begin{cases} \Delta u + f(u,v) = 0 \\ \Delta v + g(u,v) = 0 \end{cases} \tag{4}$$

1. Let $f(u,v) = -\frac{1}{\beta}v$, $g(u,v) = \alpha u$. We have:

$$\begin{cases} \Delta u - \frac{1}{\beta}v = 0 \\ \Delta v + \alpha u = 0 \end{cases} \tag{5}$$

The functional corresponding to this system is $\varphi(u,v) = \alpha u^2 + \beta v^2$. Hence, since $\alpha \geq 0$, $\beta > 0$, φ satisfies Theorem 1.1. We have:

Theorem 2.1. *If (u,v) is a solution of (5) and $\alpha \geq 0$, $\beta > 0$, then $\alpha u^2 + \beta v^2$ verifies MP.*

Remark 2.1. This result represent a generalization of example 1, given in [1].

2. Let $f(u,v) = -\alpha u - \beta v,\ g(u,v) = \delta u + \gamma v$.We have:

$$\begin{cases} \Delta u - \alpha u - \beta v = 0 \\ \Delta v + \delta u + \gamma v = 0 \end{cases} \tag{6}$$

The equation corresponding to this system is:

$$\frac{du}{-\alpha u - \beta v} = \frac{dv}{\delta u + \gamma v}$$

If $u = zv$, we obtain:

$$\frac{\gamma z + \delta}{\gamma z^2 + (\alpha + \delta)z + \beta} = -\frac{1}{v}dv$$

and if we put $\int \frac{\gamma z + \delta}{\gamma z^2 + (\alpha + \delta)z + \beta}dz = \ln F(z)$, we will have:

$$\varphi(u, v) = \Phi\left[vF\left(\frac{u}{v}\right)\right], \Phi \in C^1(\mathbb{R})$$

We can consider $\varphi(u, v) = vF\left(\frac{u}{v}\right)$, but because of F, the properties of such functional are very hard to study.

What we can observe is that if $\alpha = \delta$ we have:

$$\frac{\gamma z + \alpha}{\gamma z^2 + 2\alpha z + \beta}dz = -\frac{1}{v}dv$$

In this way we will obtain:

$$\varphi(u, v) = \Phi\left(\sqrt{\gamma u^2 + \alpha uv + \beta v^2}\right)$$

where $\Phi \in C^1(\mathbb{R})$.

If we put $\Phi(t) = t^2$, then:

$$\varphi(u, v) = \gamma u^2 + \alpha uv + \beta v^2.$$

Theorem 2.2. *If (u,v) is a solution of (6), and the matrix* $\begin{pmatrix} 2\gamma & \alpha \\ \alpha & 2\beta \end{pmatrix}$ *is positive semidefinite i.e.* $\gamma \geq 0$, $\alpha^2 \leq 4\beta\gamma$, *then* $\gamma u^2 + \alpha uv + \beta v^2$ *verifies MP.*

Remark 2.2. This result represents a generalization of 1.

3. In the general case of system (4)

$$\begin{cases} \Delta u + f(u, v) = 0 \\ \Delta v + g(u, v) = 0 \end{cases}$$

the corresponding equation is $\frac{du}{dv} = \frac{f(u, v)}{g(u, v)}$:

$$g(u, v)du - f(u, v)dv = 0. \tag{7}$$

We consider the differential form $\omega = g(u,v)du - f(u,v)dv$. ω is a total differential if:

$$\frac{\partial g}{\partial v} = -\frac{\partial f}{\partial u}. \tag{8}$$

We will choose φ in the form:

$$\varphi(u,v) = \int_{(0,0)}^{(u,v)} g(u,v)du - f(u,v)dv + C \tag{9}$$

The conditions of Theorem 1.1, becomes:

$$g(u,v)\int_0^u \frac{\partial g(u,v)}{\partial v}du - f(u,v)\int_0^v \frac{\partial f(u,v)}{\partial u}dv \le 0 \tag{10}$$

$$\frac{\partial g}{\partial u} - \int_0^v \frac{\partial f^2(u,v)}{\partial u^2}dv \ge 0 \tag{11}$$

$$\left(\frac{\partial g}{\partial u} - \int_0^v \frac{\partial^2 f(u,v)}{\partial u^2}dv\right)\left(\int_0^u \frac{\partial^2 g(u,v)}{\partial v^2}du - \frac{\partial f}{\partial v}\right) \ge \left(\frac{\partial g}{\partial v} - \frac{\partial f}{\partial u}\right)^2 \tag{12}$$

Because of (8) we have:

$$-\frac{\partial^2 f}{\partial u^2} = \frac{\partial^2 g}{\partial u \partial v};\ \frac{\partial^2 g}{\partial v^2} = -\frac{\partial^2 f}{\partial u \partial v};\ -\int_0^v \frac{\partial^2 f}{\partial u^2}dv = \frac{\partial g}{\partial u};\ \int_0^u \frac{\partial^2 g}{\partial v^2}du = -\frac{\partial f}{\partial v}$$

In these conditions, (10), (11), (12), becomes

$$0 \le 0$$

$$\frac{\partial g}{\partial u} \ge 0 \tag{13}$$

$$\left(\frac{\partial g}{\partial v}\right)^2 \le -\frac{\partial g}{\partial u}\frac{\partial f}{\partial v} \tag{14}$$

$$\left(\frac{\partial f}{\partial u}\right)^2 \le -\frac{\partial g}{\partial u}\frac{\partial f}{\partial v} \tag{15}$$

It is obvious that (14) or (15) are satisfied if:

$$\frac{\partial f}{\partial v} \le 0. \tag{16}$$

Theorem 2.3. *In conditions (8), (13), (14/15), (16), if (u,v) is a solution of (4), then (9) verifies MP.*

Remark 2.3. If $f(u,v) = f(v)$, $g(u,v) = g(u)$, the conditions from above are: $g'(u) \ge 0$, $f'(v) \le 0$. This case appears in [1].

As an example if $f(u,v) = u - 2v,\ g(u,v) = 2u - v$, then the conditions of Theorem 2.3 are satisfied and this implies that $(u-v)^2$ verifies MP.

Let us consider now the functions from 2, i.e.

$$f(u,v) = -\alpha u - \beta v$$
$$g(u,v) = \gamma u + \delta v$$

From (8) we have $\delta = \alpha$, and so g is $g(u,v) = \gamma u + \alpha v$.

Conditions (13), (14/15), (16) are: $\gamma \geq 0$, $\beta \leq 0$, $\alpha^2 \leq \beta\gamma$. In conclusion if (u,v) is a solution of (4), with f and g as above, and $\gamma \geq 0$, $\beta \leq 0$, $\alpha^2 \leq \beta\gamma$, then $\frac{1}{2}\gamma u^2 + 2\alpha uv + \frac{1}{2}\beta v^2$ verifies MP.

Remark 2.4. In this way (but choosing another method) we have obtained a functional as in 2, and the condition are the same.

4. Let us consider the system

$$\begin{cases} -\Delta u = \lambda f(x,u) - v \\ -\Delta v = \delta u - \gamma v \end{cases} \tag{17}$$

This system appears in [2] and the authors are looking for the existence of a positive solution. We will try to find a functional with the properties from Theorem 1.1.

Let $f(x,u) = f(u)$. We have:

$$\begin{cases} -\Delta u = \lambda f(u) - v \\ -\Delta v = \delta u - \gamma v \end{cases} \tag{18}$$

We will put $f_1(u,v) = \lambda f(u) - v$, $g_1(u,v) = \delta u - \gamma v$, and obtain:

$$\begin{cases} \Delta u + f_1(u,v) = 0 \\ \Delta v + g_1(u,v) = 0 \end{cases} \tag{19}$$

From (8) we have $-\gamma = -\lambda f'(u)$, *i.e.* $f'(u) = \frac{\gamma}{\lambda} u$

(19) becomes:

$$\begin{cases} \Delta u + \gamma u - v = 0 \\ \Delta v + \delta u - \gamma v = 0 \end{cases} \tag{20}$$

The conditions (13), (14/15),(16), are satisfied if $\delta \geq 0$, $\gamma^2 \leq \delta$.

So, if $\delta \geq 0$, $\gamma^2 \leq \delta$, , and (u,v) is a solution of (20), then $\frac{1}{2}\delta u^2 + 2\gamma uv + \frac{1}{2}v^2$, verifies MP.

Remark 2.5. This is a particular case of example given at 3.

Remark 2.6. If we try to find φ, in the classical way, we´ll obtain $\delta u^2 + 2\gamma uv + v^2$, which, in the same conditions, verifies MP.

Remark 2.7. We can try to find a integrating factor for

$$(\delta u - \gamma v)du + (v - \lambda f(u))dv = 0$$

from:

$$(\delta u - \gamma v)\frac{\partial \mu}{\partial v} - (v - \lambda f(u))\frac{\partial \mu}{\partial u} + (-\gamma + \lambda f\'(u))\mu = 0.$$

Let us consider now the system:

$$\begin{cases} \Delta u + f(u,v,w) = 0 \\ \Delta v + g(u,v,w) = \\ \Delta w + h(u,v,w) = 0 \end{cases} \tag{21}$$

5. Let $f(u,v,w) = -v - w$, $g(u,v,w) = u - w$, $h(u,v,w) = u + v$, (21) becomes:

$$\begin{cases} \Delta u - v - w = 0 \\ \Delta v + u - w = 0 \\ \Delta w + u + v = 0 \end{cases} \tag{22}$$

Let $\varphi(u,v) = u^2 + v^2 + w^2$. Condition (ii) from Theorem 1.1 becomes:

$$\frac{\partial \varphi}{\partial u} f(u,v,w) + \frac{\partial \varphi}{\partial v} g(u,v,w) + \frac{\partial \varphi}{\partial w} h(u,v,w) \leq 0.$$

φ satisfies this condition, and the hessian of φ is $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ which is positive definite. We have the following result:

Theorem 2.4. *If (u,v,w) is a solution of (22) then* $u^2 + v^2 + w^2$ *verifies MP.*

6. Let $f(u,v,w) = -\beta v - \gamma w$, $g(u,v,w) = \alpha u - \gamma w$, $h(u,v,w) = \alpha u + \beta v$, (21) becomes:

$$\begin{cases} \Delta u - \beta v - \gamma w = 0 \\ \Delta v + \alpha u - \gamma w = 0 \\ \Delta w + \alpha u + \beta v = 0 \end{cases} \tag{23}$$

Let $\varphi(u,v) = \alpha u^2 + \beta v^2 + \gamma w^2$. Condition (ii) from Theorem 1.1 is verified by φ. The hessian of φ is $\begin{pmatrix} 2\alpha & 0 & 0 \\ 0 & 2\beta & 0 \\ 0 & 0 & 2\gamma \end{pmatrix}$ which is positive definite if $\alpha, \beta, \gamma \geq 0$.

Theorem 2.5. *If (u,v,w) is a solution of (24), and $\alpha, \beta, \gamma \geq 0$, then $\alpha u^2 + \beta v^2 + \gamma w^2$ verifies MP.*

7. Let $f(u,v,w) = w-v$, $g(u,v,w) = u-w$, $h(u,v,w) = v-u$, (21) becomes:

$$\begin{cases} \Delta u + w - v = 0 \\ \Delta v + u - w = 0 \\ \Delta w + v - u = 0 \end{cases} \tag{24}$$

Let $\varphi_1(u,v,w) = u^2 + v^2 + w^2$. (ii) from Theorem 1.1 is verified by φ, and the hessian of φ is $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ which is positive definite.

Theorem 2.6. *If (u,v,w) is a solution of (24) then $u^2+v^2+w^2$ verifies MP.*

Let now $\varphi_2(u,v) = u^2 + v^2 + w^2 + uv + uw + vw$. φ verifies the condition (ii) from Theorem 1.1. The hessian of φ is:

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{7}{6} \end{pmatrix}$$

and it is positive definite. In this way we obtain the following result:

Theorem 2.7. *If (u,v,w) is a solution of (230 then $u^2+v^2+w^2+uv+uw+vw$ verifies MP.*

Remark 2.8. It is obvious that the example from above prove the fact that the functional corresponding to a system, and which satisfy am maximum principle, is not unique.

8. Let $f(u,v,w) = -u+v-w$, $g(u,v,w) = -u-v+w$, $h(u,v,w) = u-v-w$, (21) becomes:

$$\begin{cases} \Delta u - u + v - w = 0 \\ \Delta v - u - v + w = 0 \\ \Delta w + u - v - w = 0 \end{cases} \tag{25}$$

Let $\varphi(u,v) = u^2+v^2+w^2$. (ii) becomes $-2(u^2+v^2+w^2) \leq 0$, and the hessian of φ, as we saw, is positive definite. We have:

Theorem 2.8. *If (u,v,w) is a solution of (25) then* $u^2+v^2+w^2$ *verifies MP.*

Remark 2.9. If $f(u,v,w) = -\alpha u + \beta v - \gamma w$, $g(u,v,w) = -\alpha u - \beta v + \gamma w$, $h(u,v,w) = \alpha u - \beta v - \gamma w$, with $\alpha,\beta,\gamma \geq 0$, and (u,v,w) is a solution of the corresponding system, then $\alpha u^2 + \beta v^2 + \gamma w^2$ verifies MP.

Let us suppose now that $c \neq 0$. Condition (ii) from Theorem 1.1 becomes:

$$-\frac{\partial\varphi}{\partial y_1}f_1 - ... - \frac{\partial\varphi}{\partial y_m}f_m + c\left(\varphi - \frac{\partial\varphi}{\partial y_1}y_1 - ... - \frac{\partial\varphi}{\partial y_m}y_m\right) \geq 0$$

$$(f_1+cy_1)\frac{\partial\varphi}{\partial y_1} + ... + (f_m+cy_m)\frac{\partial\varphi}{\partial y_m} \leq c\varphi$$

Let $\varphi = y_1^2 + ... + y_m^2$. We have:

$$2y_1f_1 + ... + 2y_mf_m + c\left(y_1^2 + ... + y_m^2\right) \leq 0 \tag{26}$$

If $m = 2$ then (26) becomes:

$$2uf(u,v) + 2vg(u,v) + c(u^2+v^2) \leq 0 \tag{27}$$

Remark 2.10. If $f = cu$ and $g = cv$, condition (27) is verified for $c \leq 0$, and so u^2+v^2 verifies MP.

Remark 2.11. If $f(u,v) = -\alpha u + \beta v$, $g(u,v) = -\beta u - \gamma v$, then condition (27) is verified for $\alpha,\beta \geq 0$, and $c \leq 0$

Remark 2.12. In general case if $c \leq 0$ and $tf(t_1,...,t_{k-1},t,t_{k+1},...,t_m) \leq 0$, then $\sum\limits_{i=1}^{m} u_i^2$ satisfies MP.

Remark 2.13. If we take $\varphi(u,v) = \alpha u^2 + \beta v^2 + \gamma w^2$ (for the system with f and g like in remark 11), then condition (27) take place if $c \leq 0$, $\alpha,\gamma > 0$, $\beta^2 < \alpha\gamma$.

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