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CRITICAL AND VECTOR CRITICAL SETS IN THE PLANE

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Abstract. Given a non-empty set $C \subset \mathbb{R}^2$, is C the set of critical points for some smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ or vectorial map $f : \mathbb{R}^2 \to \mathbb{R}^2$? In this paper we give some results in this direction.

1. Introduction

A point $p \in \mathbb{R}^2$ is *critical* for a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ if its derivative at p is zero. $(df)_p = 0$. This means $\dfrac{\partial f}{\partial x}(p) = \dfrac{\partial f}{\partial y}(p) = 0$, in a smooth chart in p. The set of all critical points of f is denoted by $C(f)$. The image of $C(f)$ is the set of *critical values* $B(f) = f(C(f))$. If x is not critical, then it is *regular*. We say that $C \subset \mathbb{R}^2$ is *critical* if $C = C(f)$ for some smooth $f : \mathbb{R}^2 \to \mathbb{R}$. A *proper function* has the property that $f^{-1}(K)$ is compact for all compact sets K. Equivalently, when $f : \mathbb{R}^2 \to \mathbb{R}$, $|f(x)| \to \infty$ iff $|x| \to \infty$. We say that $C \subset \mathbb{R}^2$ is *properly critical* if f can be chosen to be proper. Clearly, a critical set is closed. What other properties does it have? In the compact case, there is just one other requirement.

Theorem. [No-Pu] *Let C be a compact non-empty subset of $\mathbb{R}^2$. The following assertions are equivalent:*

1. *C is critical*
2. *C is properly critical*
3. *The components of its complement are multiply connected.*

A *component* of a topological space is a maximal connected subset of the space. It is *multiply connected* if it is not simply connected. The condition on multiply connectivity is a topological condition on the complement, not on the space. If C is any finite set of points or a Cantor set in the plane, then it is properly critical. Their complements are multiply connected. On the other hand, a circle is not critical. If C

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is the union of a circle and a point, then it is critical if and only if the point is inside the circle.

If its critical set is noncompact, it is unreasonable to expect properness of f. If $C = C(f)$ is closed, unbounded and connected, then by Sard's theorem, f is constant on C, $f(C) = c$, and $f^{-1}(c)$ is noncompact, so f is not proper.

Theorem. *If $C \subset \mathbb{R}^2$ is critical, compact and non-empty, then any bounded component of its complement has disconnected boundary. In particular, no compact curve in $\mathbb{R}^2$, smooth or not, is a critical set.*

Given a closed, noncompact set $K \subset \mathbb{R}^2$ when is there a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $K = C(f)$? We say that ∞ is *arcwise accessible* in $U \subset \mathbb{R}^2$ if there is an arc $\alpha : [0, \infty) \to U$ such that $\alpha(t) \to \infty$ as $t \to \infty$.

Theorem. [No-Pu] *A closed set $K \subset \mathbb{R}^2$ is critical if and only if ∞ is arcwise accessible in each simply connected component of $\mathbb{R}^2 \setminus K$.*

2. **Vector critical sets**

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth map. The point $p \in \mathbb{R}^2$ is a *critical point* of f if $rank_p f \leq 1$. If f is given by $f = (f_1, f_2)$, then in some local chart around p, p is critical point of f if and only if the Jacobi matrix of f in p is singular, which means:

$$\det \begin{bmatrix} \dfrac{\partial f_1}{\partial x}(x_0, y_0) & \dfrac{\partial f_1}{\partial y}(x_0, y_0) \\ \dfrac{\partial f_2}{\partial x}(x_0, y_0) & \dfrac{\partial f_2}{\partial y}(x_0, y_0) \end{bmatrix} = 0$$

The set $C \subseteq \mathbb{R}^2$ is *vector critical* if it is the critical set of some smooth map $f : \mathbb{R}^2 \to \mathbb{R}^2$. In which conditions will a critical set $C \subset \mathbb{R}^2$ be vector critical? For a class of subsets of the plane, the answer is given by the following theorem:

Theorem. *Any critical set $C \subset \mathbb{R}^2$ is vector critical.*

Proof: Since C is critical, there is a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$, so that $C = C(f)$, where

$$C(f) = \left\{ (x_0, y_0) \in \mathbb{R}^2 : \frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0 \right\}.$$

Define $F : \mathbb{R}^2 \to \mathbb{R}^2$, by $F(x, y) = (h(x, y), y)$, where $h : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$h(x,y) = \int_0^x \Big(\Big[\frac{\partial f}{\partial x}(x,y)\Big]^2 + \Big[\frac{\partial f}{\partial y}(x,y)\Big]^2\Big) dx.$$

Since h is smooth, so is F. We show that $C(f) = C(F)$.

The Jacobi matrix of f in some point $(x_0, y_0) \in \mathbb{R}^2$ is

$$J(F)(x_0,y_0) = \begin{bmatrix} \Big[\frac{\partial f}{\partial x}(x_0,y_0)\Big]^2 + \Big[\frac{\partial f}{\partial y}(x_0,y_0)\Big]^2 & \frac{\partial h}{\partial y}(x_0,y_0) \\ 0 & 1 \end{bmatrix}.$$

For $(x_0, y_0) \in C(f)$, we have $\frac{\partial f}{\partial x}(x_0,y_0) = \frac{\partial f}{\partial y}(x_0,y_0) = 0$, so

$$J(F)(x_0,y_0) = \begin{bmatrix} 0 & \frac{\partial h}{\partial y}(x_0,y_0) \\ 0 & 1 \end{bmatrix}$$

and $(x_0, y_0) \in C(F)$. Conversely, if $(x_0, y_0) \in C(F)$, it follows that $\Big[\frac{\partial f}{\partial x}(x_0,y_0)\Big]^2 + \Big[\frac{\partial f}{\partial y}(x_0,y_0)\Big]^2 = 0$, and then $\frac{\partial f}{\partial x}(x_0,y_0) = \frac{\partial f}{\partial y}(x_0,y_0) = 0$, so $(x_0, y_0) \in C(f)$. $\square$

If, in theorem above f is supposed to be a harmonic function (this means that f has the property $\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = 0$), then F could be defined to be the map $F = (f, g)$, where $g : \mathbb{R}^2 \to \mathbb{R}$ is the smooth map which is the solution of the system

$$\begin{cases} \frac{\partial g}{\partial x}(x,y) &= -\frac{\partial f}{\partial y}(x,y) \\ \frac{\partial g}{\partial y}(x,y) &= \frac{\partial f}{\partial x}(x,y). \end{cases}$$

The converse of this theorem is not true. There are more vector critical sets than critical. A vector critical set which is not critical is the circle in the plane. The map $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) = (\frac{x^3}{3} + xy - x, y)$ is critical exactly on the unit circle in $\mathbb{R}^2$.

3. The family of excellent mappings

An *excellent mapping* is a smooth function $f : \mathbb{R}^2 \to \mathbb{R}^2$ whose critical points are all folds or cusps. A *fold* is a critical point such that, after smooth local changes of coordinates in the domain and image, the function is of the form

$$f(x, y) = (x^2, y),$$

the critical point being taken to the origin. For a *cusp*, after a change of coordinates, the function is of the form

$$f(x, y) = (xy - x^3, y),$$

where the critical point is taken to the origin.

For an excellent mapping, the set of critical points will consist of smooth curves; we call these *general folds* of the mapping. Also, the cusp points are isolated on the general fold. Let f be an excellent mapping and C a general fold of f through p. Thus p will be a fold point if the image of C near p is a smooth curve with non-zero tangent vector at p, and p will be a cusp point if the tangent vector is zero at p but it becoming non-zero at a positive rate as we move away from p on C.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an excellent mapping. The *derivative* of f with respect to V at p is the vector in $\mathbb{R}^2$

$$\nabla_V f(p) = \lim_{t \to 0_+} \frac{1}{t}[f(p + tV) - f(p)].$$

For each $p \in \mathbb{R}^2$, consider the vectors $V' = \nabla_V f(p)$ as a function of vectors V with $|V| = 1$. We shall use a certain system of curves defined by f in an open set $R \subset \mathbb{R}^2$. We let R contain p if the vectors V' are not all of the same length. For any $p \in R$, there will be a pair of opposite directions at p such that for V in these directions,

$|V'|$ is a minimum. (For V in the perpendicular direction, $|V'|$ will be a maximum.) Now R is filled up by smooth curves in these directions; we call these curves *curves of minimum* ∇f.

For any $p \in R$ and vector $V \neq 0$, $\nabla_V f(p) = 0$ if and only if p is a singular point and V is tangent to the curve of minimum ∇f.

Consider any general fold curve C. If a curve of minimum ∇f cuts C at a positive angle at p, then for the tangent vector $V(p)$, $\nabla_V f(p) \neq 0$, and hence p is a fold point. Suppose C is tangent to a curve of minimum ∇f at p. Then p is not a fold point, and hence is a cusp point, since f is excellent. Set $V^* = \nabla_V \nabla_V f(p)$; then $V^* \neq 0$. Since $\nabla_V f(p) = 0$, $\nabla_v f(p')$ is approximately in the direction of $\pm V^*$ for p' on C near p. It follows that $\nabla_W f(p)$ is a multiple of V^*, for all vectors W. As we move along the general fold through p, $\nabla_V f(p')$ changes from a negative to a positive multiple of V^* (approximately); hence $V(p')$ cutes the curves of minimum ∇f in opposite senses on the two sides of p. Therefore the curves of minimum ∇f lying on one side of C cut C on both sides of p. We call this side of C the *upper side* and the other the *lower side*.

The image of C has a cusp at $f(p)$, pointing in the direction of $-V^*$. For any vector W not tangent to C at p, $\nabla_W f(p)$ is a positive or negative multiple of V^*, according as W points into the upper or lower side of C.

Let f and g be mappings $\mathbb{R}^2 \to \mathbb{R}^2$ and $\varepsilon(p)$ a positive continuous function in $\mathbb{R}$. We say g is an *ε-approximation* to f if

$$|g(p) - f(p)| < \varepsilon(p), \quad \forall p \in \mathbb{R}.$$

If f and g are r-smooth, we say g is an *(r, ε)-approximation* to f if this inequality holds, and also the similar inequalities for all partial derivatives of orders $\leq r$, using fixed coordinate systems. We speak of *general approximations* and *r-approximations* in the two cases.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an excellent mapping. We describe certain approximations g to f which have the singularities of f and also further singularities.

(a) *Arbitrary approximations:* For any smooth curve C in the plane which touches no general fold, we may introduce two new folds, one at C and one near C.

For each $p \in C$, let p_t, $-1 \leq t \leq 1$, denote the points of a line segment S_p approximately perpendicular to C in p, with $p_0 = p$. We may choose these segments so that they cover a neighborhood U of C which touches no general fold of f. We change f to obtain g as follows: as t runs from -1 to 1, let $g(p_t)$ run along $f(S_p)$ from $f(p_{-1})$ to $f(p)$, then back a little, then on through $f(p)$ to $f(p_1)$. If f and C are smooth, we may construct g to be smooth. C is a fold for g and so is a curve C', consisting of the points $p_{1/2}$, for example. We may let $g = f$ in $\mathbb{R}^2 \setminus U$. With U small enough, g is an arbitrarily good approximation of f.

(b)*Approximations with first derivatives:* Let C_0 be a curve of fold points of f, without cusps. It may be the whole or a part of a complete general fold of f. We show that we may define g to be an arbitrarily good approximation of f together with first derivatives, so that there is a new pair of folds near C_0. If C_0 is closed, there will be no new cusps for g; otherwise, the new folds will meet in a pair of cusp points for g.

We may let p_t denote points of a neighborhood of C_0, as in (a), so that the image of each S_p under f is an arc folded over on itself, the fold occurring at p. Let $g(p_t) = f(p_t)$ for $-1 \leq t \leq 0$; as t runs from 0 to 1, let $g(p_t)$ move along $f(S_p)$ towards $f(p_1)$, then back a little, and then forward again to $f(p_1)$. So, we obtain two new folds.

We show that we may make g approximate to f near a given point p of C_0. Then, the approximation is possible near the all of C_0.

We may choose the coordinates so that f, near p, is given by

$$f(x, y) = (x^2, y).$$

We may define a smooth function $\phi : \mathbb{R} \to \mathbb{R}$, so that:

1. $\phi(-t) = \phi(t)$, for all $t \in \mathbb{R}$
2. $\phi(0) = 1$
3. $\phi(t) = 0$, for $|t| \geq 1$
4. $0 \leq \phi'(t) \leq \phi'(-\frac{1}{2}) = \alpha$, for $t < 0$.

For $\varepsilon > 0$, define $g : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$g(x,y) = \left(x^2 + \frac{10\varepsilon^2}{\alpha}\phi\left(\frac{x-2\varepsilon}{\varepsilon}\right), y \right).$$

g is smooth and the Jacobian matrix of g has the form

$$J(g)(x,y) = \begin{bmatrix} 2x + \frac{10\varepsilon}{\alpha}\cdot\phi'\left(\frac{x-2\varepsilon}{\varepsilon}\right) & 0 \\ 0 & 1 \end{bmatrix}$$

For $x \in (-\infty, \varepsilon] \cup [3\varepsilon, \infty)$, $\phi\left(\frac{x-2\varepsilon}{\varepsilon}\right) = 0$. So, p is also a critical point of g. Moreover, as

$$\det J(g)(2\varepsilon, y) = 4\varepsilon + \frac{10\varepsilon}{\alpha}\cdot\phi'(0) = 4\varepsilon > 0$$

$$\det J(g)\left(\frac{5\varepsilon}{2}, y\right) = 5\varepsilon + \frac{10\varepsilon}{\alpha}\cdot\phi'\left(\frac{1}{2}\right) = 5\varepsilon + \frac{10\varepsilon}{\alpha}\cdot(-\alpha) = -5\varepsilon < 0$$

$$\det J(g)(3\varepsilon, y) = 6\varepsilon + \frac{10\varepsilon}{\alpha}\cdot\phi'(1) = 6\varepsilon > 0,$$

then there are two numbers $x_1 \in (2\varepsilon, \frac{5\varepsilon}{2})$ and $x_2 \in (\frac{5\varepsilon}{2}, 3\varepsilon)$, so that

$$\det J(g)(x_1, y) = \det J(g)(x_2, y) = 0 :$$

these define the points of the new folds.

Also, g is an approximation of f with first derivatives:

$$\left| 2x + \frac{10\varepsilon}{\alpha}\phi'\left(\frac{x-2\varepsilon}{\varepsilon}\right) - 2x \right| = \left| \frac{10\varepsilon}{\alpha}\phi'\left(\frac{x-2\varepsilon}{\varepsilon}\right) \right| \le 10\varepsilon, \quad \forall x \in \mathbb{R}.$$

We show now how we may insert cusps. We consider several types of approximation.

(a)*Arbitrarily approximation:* We show that we may insert a pair of nearby arcs where the new function g will have fold points and run them together to give the new cusps.

We consider the smooth curve C, which touches no general fold of f and $p \in C$, as before. Suppose that near the regular point p, f is given by $f(x, y) = (x, y)$. Define ϕ as before and define $g : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$g(x, y) = \left(x + \frac{2\varepsilon}{\alpha} \phi \left(\frac{x}{\varepsilon} \right) \phi \left(\frac{y}{\varepsilon} \right), y \right).$$

Then g is smooth, is an arbitrarily good approximation of f and $g = f$ outside a small neighborhood of p. The critical points of g are those of f and those given by

$$\det J(g)(x, y) = \det \begin{bmatrix} 1 + \dfrac{2}{\alpha} \phi \left(\dfrac{y}{\varepsilon} \right) \phi' \left(\dfrac{x}{\varepsilon} \right) & \dfrac{2}{\alpha} \phi \left(\dfrac{x}{\varepsilon} \right) \phi' \left(\dfrac{y}{\varepsilon} \right) \\ 0 & 1 \end{bmatrix} = 0,$$

or

$$1 + \frac{2}{\alpha} \phi \left(\frac{y}{\varepsilon} \right) \phi' \left(\frac{x}{\varepsilon} \right) = 0.$$

Since $\det J(g)(0, 0) = 1 > 0$, $\det J(g) \left(\dfrac{\varepsilon}{2}, 0 \right) = 1 + \dfrac{2}{\alpha} \cdot 1 \cdot (-\alpha) = -1 < 0$, and $\det J(g)(2\varepsilon, 0) = 1 > 0$, it is clear that there are two folds cutting the x-axis. If ϕ is sufficiently simple shape, these come together in two cusps.

(b)*Approximations with first derivatives:* Let p be a fold point of f, on a critical curve of f which contains no cusp points. Near p, f is given by $f(x, y) = (x^2, y)$. We define $g : \mathbb{R}^2 \to \mathbb{R}^2$, by

$$g(x, y) = \left(x^2 + \frac{10\varepsilon^2}{\alpha} \phi \left(\frac{x - 2\varepsilon}{\varepsilon} \right) \phi \left(\frac{y}{\varepsilon} \right) \phi \left(\frac{y}{\varepsilon} \right), y \right),$$

with ϕ chosen as before. Outside a little neighborhood of p, $g = f$. We have

$$J(g)(x, y) = \begin{bmatrix} 2x + \dfrac{10\varepsilon}{\alpha} \phi' \left(\dfrac{x - 2\varepsilon}{\varepsilon} \right) \phi \left(\dfrac{y}{\varepsilon} \right) & \dfrac{\partial g}{\partial y}(x, y) \\ 0 & 1 \end{bmatrix},$$

so $\det J(g)(0, 0) = \dfrac{10\varepsilon}{\alpha} \phi'(-2)\phi(0) = 0$, which means p is a critical point of g. Since

$$\det J(g)(2\varepsilon, 0) = 4\varepsilon + \frac{10\varepsilon}{\alpha} \phi'(0)\phi(0) = 4\varepsilon > 0$$

$\det J(g)\left(\frac{5\varepsilon}{2}, 0\right) = 5\varepsilon + \frac{10\varepsilon}{\alpha}(-\alpha)\phi(0) = -5\varepsilon < 0$

$\det J(g)(3\varepsilon, 0) = 6\varepsilon + \frac{10\varepsilon}{\alpha}\phi'(1)\phi(0) = 6\varepsilon > 0,$

$\det J(g)$ becomes zero for two points of the x-axis. We obtain two new folds, joined at two cusp points, and g is an arbitrarily good approximation of f, together with first derivatives:

$$\left|2x + \frac{10\varepsilon}{\alpha}\phi'\left(\frac{x-2\varepsilon}{\varepsilon}\right)\phi\left(\frac{y}{\varepsilon}\right) - 2x\right| = \left|\frac{10\varepsilon}{\alpha}\phi'\left(\frac{x-2\varepsilon}{\varepsilon}\right)\phi\left(\frac{y}{\varepsilon}\right)\right| <$$

$$< \frac{10\varepsilon}{\alpha}\cdot\alpha\cdot 1 = 10\varepsilon, \quad \forall\,(x,y)\in\mathbb{R}^2.$$

(c) *Approximations with first and second derivatives:* Let p be a cusp point of f. Near p, f is given by $f(x,y) = (xy - x^3, y)$. Define g near p by setting

$$g(x,y) = \left(xy - x^3\left[1 - 2\phi\left(\frac{x}{\varepsilon}\right)\phi\left(\frac{y}{\varepsilon}\right)\right], y\right).$$

Then

$$J(g)(x,y) = \begin{bmatrix} y - 3x^2\left[1 - 2\phi\left(\frac{x}{\varepsilon}\right)\phi\left(\frac{y}{\varepsilon}\right)\right] + 2x^3\cdot\frac{1}{\varepsilon}\phi'\left(\frac{x}{\varepsilon}\right)\phi\left(\frac{y}{\varepsilon}\right) & \frac{\partial g}{\partial y}(x,y) \\ 0 & 1 \end{bmatrix}$$

The curve C of general fold of g coincides with the original critical curve C_0: $y = 3x^2$ of f for $|x| \geq \varepsilon$, it contains p and, by symmetry, is in the x-direction. Since

$$\frac{\partial f_1}{\partial x}(p) = \frac{\partial f_1}{\partial y}(p) = 0, \quad \frac{\partial^2 f_1}{\partial x^2}\partial x\partial y(p) = 1 \quad \text{şi} \quad \frac{\partial^3 f_1}{\partial x^3}(p) = 6,$$

p is a cusp point for g [Wh]. At points of C where $x \leq -\varepsilon$, $g = f$ and $J(g)(x,y)\begin{bmatrix} y-3x^2 & x \\ 0 & 1\end{bmatrix}$, so $\frac{\partial^2 f_1}{\partial x^2}(x,y) = -6x > 0$. For $x \geq \varepsilon$, $g = f$ and $\frac{\partial^2 f_1}{\partial x^2}(x,y) = -6x < 0$. On the other hand, since $\frac{\partial^2 f_1}{\partial x^2}(p) = 0$ and $\frac{\partial^3 f_1}{\partial x^3}(p) > 0$, we have that $\frac{\partial^2 f_1}{\partial x^2}(x,y)$ has the same sign as x for $x \neq 0$ and $|x|$ small enough.

Therefore, as x runs from $-\varepsilon$ to ε, if we run along C, $\dfrac{\partial J}{\partial x} = \dfrac{\partial^2 f_1}{\partial x^2}$ changes sign at least three times. With the function ϕ of simple shape, it will change sign exactly three times; that is g will have three cusp points. We have thus introduced two new cusps, the three cusps lying on a single general fold curve.

Differentiating g, it follows that g is an arbitrarily good approximation of f, together with first and second derivatives.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an excellent mapping and p a cusp point on the general fold C. Suppose there is a smooth curve A which moves from p to ∞ into the lower side of C and which touches no general fold. Then there is arbitrarily good approximation g to f which agrees with f outside a neighborhood U of A, and for which the part of the fold near p is replaced by a pair of folds going near A, to ∞, without cusp points.

This may be seen as follows. Around p, f is given by $f(x, y) = (xy - x^3, y)$. Each line $y = a > 0$ is mapped by f so as to fold over on itself twice. The lines $y = a \leq 0$ have no such folds. We need merely insert such folds near the negative y-axis, to join the above folds. These can be extended down along all of A.

We saw that cusps may be eliminated from regions by arbitrarily good approximations. This is not true for folds.

Theorem 3.1. *Let p be a fold point of the excellent mapping f. Then for any neighborhood U of p, each sufficiently good approximation g to f which is excellent has a fold point in U.*

Proof: Since p is a fold point, there are two points p_1 and p_2 in U where the Jacobian has opposite signs. Let U_i be a circular neighborhood of p_i $(i = 1, 2)$ which touches no fold, and let U_i' be an interior circular neighborhood. For a sufficiently good approximation g to f, if g_t is the deformation of g into f,

$$g_t(q) = g(q) + t[f(q) - g(q)] \qquad (0 \leq t \leq 1),$$

then the image of the boundary ∂U_i does not touch the image of U_i' under f:

$$g_t(q) \neq f(q'), \quad q \in \partial U_i, \quad q' \in U_i', \quad 0 \leq t \leq 1.$$

Hence $g(U_i)$ and $f(U_i)$ cover $f(U_i')$ the same algebraic number of times. For f, this number is ± 1. Hence there is a point p_i' in U_i' such that the Jacobian of g at p_i' is of the same sign as the Jacobian of f in U_i. But the Jacobians of g at p_1' and at p_2' are of opposite sign. Then the segment $p_1'\, p_2'$ contains a singular point of g, and since g is excellent, there is a fold point of g in U. □

Theorem 3.2. *If Q is a bounded closed set in which f is non-singular, then any sufficiently good approximation g to f is non-singular in Q.*

Proof: It follows since the Jacobian involves only first derivatives. □

Theorem 3.3. *Let the arc A have end points p_1 and p_2 where f is non-singular. Then, for any sufficiently good 1-approximation g to f which is excellent, any arc A' from p_1 to p_2 which cuts only fold points of f and g cuts the same number of folds (mod 2) for each.*

Proof: This is clear, since the Jacobian of f and of g have the same sign at each p_i. □

Theorem 3.4. *Let p be a cusp point of f. Then for any neighborhood U of p, each sufficiently good 1-approximation g of f which is excellent has a cusp point in U.*

Proof: There is a curve $A = p_1\, p_2\, p_3\, p_4$ of minimum ∇f in U, which cuts the fold C through p at the points p_2 and p_3. The open arc $p_2\, p_3$ lies in the upper part of C and the open arcs $p_1\, p_2$ and $p_3\, p_4$ lie in the lower part. There is an arc B from p_1 to p_4 in the lower part of C, lying in U, such that A and B bound a region R' filled by curves of minimum ∇f. For any sufficiently good 1-approximation g to f, there will be an arc A^* of minimum ∇g, near A, which will bound, with part of B, a region R^* filled by curves of minimum ∇g. Also, g will be non-singular in B, and there will be fold points of g in R^*. The set Q of fold and cusp points of g in the closure $\overline{R^*}$ is a closed set. There is a lowest curve D of minimum ∇g in $\overline{R^*}$ which touches Q in a point p^*. Since p^* is not in B, $p^* \in R^*$. p^* is a singular point of g. Also, by definition of D, the general fold of g through p^* does not cross the curve D, and hence is tangent to D. Therefore, p^* is not a fold point of g and it follows that p^* is a cusp point of g. □

Theorem 3.5. *For any bounded closed set Q in which the only singularities of f are fold points, any sufficiently good 2-approximation g of f which is excellent has only folds in Q.*

Proof: Let p be a fold point of f in Q and let A be a short segment perpendicular to the fold, centered at p. Since $J(f)$ is of opposite signs at the two ends of A, so $F(g)$ will be. Hence $J(g)$ will vanish somewhere on A. Since f is excellent, the directional derivative of $J(f)$ in the direction of A is non-zero, hence the same is true for g and g has just one general fold cutting A. Thus the general folds of g are like those of f in Q, if the 2-approximation is good enough. Since the directions of curves of minimum ∇g and of general folds for g are nearly parallel to the similar curves for f, the conditions for fold points will be satisfied at all general fold points of g in Q, for a good approximation. Hence g will have no cusp points in Q. □

Theorem 3.6. *Let U be a neighborhood of the cusp point p of f. Then for any sufficiently good 2-approximation g to f which is excellent, there will be a cusp point p' of g in U, on a general fold C'; there will be no other general folds of g in U, and the number of critical points of g on C' in U will be odd.*

Proof: There will be a unique general fold C' of g in U. At two points p_1, p_2 of the general fold C of f, on opposite sides of p, the curves of minimum ∇f cut C in opposite senses; the same will be true, using g, for similar points p_1', p_2' of C'. Hence there will be an odd number of cusps of g between these points. There will be no cusps in $C' \cap U$ outside these points. □

Theorem 3.7. *With U, p and f as in the last theorem, any sufficiently good 3-approximation g to f has a unique general fold in U, with a unique cusp point on it.*

Proof: There is a unique C' as in the last theorem, with a cusp point p'. Since $\nabla_v \nabla_v f(p) \neq 0$, the similar relation $\nabla_{v'} \nabla_{v'} g(p') \neq 0$ holds. We see that $\nabla_{v'} g$ is in opposite directions on opposite sides of p' on C', and hence p' is the only cusp of g in U. □

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