STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 2, June 2002

CUBATURE FORMULAS ON TRIANGLE

MILENA SOLOMON

Abstract. In this article are presented some cubature formulas on triangle T which are obtained by the product of known quadrature formulas and some formulas obtaining by an approximation formula on triangle.

1. **Introduction**

Let $T = \{(x, y) \in \mathbb{R}^2 |\ x \geq 0,\ y \geq 0,\ x + y \leq 1\}$ the standard triangle from the Euclidean space $\mathbb{R}^2$, $f : T \to \mathbb{R}$ an integrable function on T, $\lambda_i f$, $i = \overline{0, m}$, some given information of f and w a nonnegative weight function on T_1.

Definition 1. The formula

$$\iint_T w(x, y)f(x, y)dxdy = \sum_{i=0}^{m} A_i \lambda_i f + R_m(f) \tag{1}$$

is called a cubature formula. The parameters A_i, $i = \overline{0, m}$, are the coefficients and $R_m(f)$ is the remainder term.

A way to construct cubature formulas on T is to use the quadrature formulas which are known from unidimensional case.

2. **Cubature formulas**

Consider

$$I(f) = \iint_T f(x, y)dxdy = \int_0^1 \int_0^{1-x} f(x, y)dxdy. \tag{2}$$

An efficient way to construct cubature formulas is to use the quadrature formulas after the integral (2) was transformed into integral on $D = [0, 1] \times [0, 1]$.

Thus, we introduce the substitution $y = t(1 - x)$ and yields

$$I(f) = \int_0^1 (1 - x) \left(\int_0^1 f(x, t(1 - x))dt \right) dx. \tag{3}$$

In order to compute the integral on $[0,1]\times[0,1]$, it can be use the product of two quadrature formulas, for example:

$$\int_0^1 g(t)dt = \sum_{i=1}^{n_t} T_i g(t_i) + R(g),$$

where $R(g)=0,\ \forall\ g\in\mathcal{P}_{d_t}$ and

$$\int_0^1 (1-x)g(x)dx = \sum_{j=1}^{n_x} A_j(1-x_j)g(x_j) + R(g),$$

where $R(g)=0,\ \forall\ g\in\mathcal{P}_{d_x}$.

Thus, we obtained an approximation on the form:

$$Q(f) = \sum_{i=1}^{n_x}\sum_{j=1}^{n_x} A_j T_i (1-x_j) f(x_j, t_i(1-x_j)). \tag{4}$$

For example, if we use the Simpson's rule:

$$\int_0^1 g(t)dt \approx \frac{1}{6}\left[g(0) + 4g\left(\frac{1}{2}\right) + g(1)\right]$$

it is obtained

$$\int_0^1 f(x,t(1-x))dt = \frac{1}{6}\left[f(x,0) + 4f\left(x,\frac{1-x}{2}\right) + f(x,1-x)\right] + R_1(f)$$

and the Simpson's rule again

$$\int_0^1 (1-x)\frac{1}{6}\left[f(x,0) + 4f\left(x,\frac{1-x}{2}\right) + f(x,1-x)\right]dx =$$

$$= \frac{1}{6}\left[\int_0^1 (1-x)f(x,0)dx + 4\int_0^1 (1-x)f\left(x,\frac{1-x}{2}\right)dx + \int_0^1 (1-x)f(x,1-x)dx\right] =$$

$$= \frac{1}{36}\left[f(0,0) + 2f\left(\frac{1}{2},0\right)\right] + \frac{4}{36}\left[f\left(0,\frac{1}{2}\right) + 4f\left(\frac{1}{2},\frac{1}{4}\right)\right] +$$

$$+\frac{1}{36}\left[f(0,1) + 2f\left(\frac{1}{2},\frac{1}{2}\right)\right] + R(f)$$

it follows

Theorem 1. *If $f\in B_{12}(0,0)$, then*

$$\iint_T f(x,y)dxdy = \frac{1}{36}[f(0,0)+f(0,1)] + \frac{1}{18}\left[f\left(\frac{1}{2},0\right) + 2f\left(0,\frac{1}{2}\right)\right] +$$

$$+\frac{2}{9}f\left(\frac{1}{2},\frac{1}{4}\right) + \frac{1}{18}f\left(\frac{1}{2},\frac{1}{2}\right) + R(f) \tag{5}$$

where

$$R(f) = \frac{1}{720} f^{(3,0)}(\xi_1, 0) + \frac{1}{32} f^{(2,1)}(\xi_2, 0) - \frac{25}{576} f^{(0,3)}(0, \eta_1) - \frac{7}{192} f^{(1,2)}(\xi_2, \eta_2)$$

with $\xi_1, \xi_2, \eta_1 \in [0,1]$ *and* $(\xi_3, \eta_3) \in T_1$.

If we use the first level, trapezoidal's quadrature

$$\int_0^1 g(t)dt = \frac{1}{2}[g(0) + g(1)] - \frac{1}{2} g''(\xi)$$

it is obtained:

$$\int_0^1 f(x, t(1-x))dt \simeq \frac{1}{2}[f(x,0) + f(x, 1-x)]$$

and, in the second level, the Simpson's quadrature:

$$\int_0^1 g(t)dt = \frac{1}{6}\left[g(0) + 4g\left(\frac{1}{2}\right) + g(1)\right] - \frac{1}{2880} f^{(4)}(\xi)$$

$$\Rightarrow \int_0^1 \frac{1-x}{2}[f(x,0) + f(x,1-x)]dx = \frac{1}{12}\left[f(0,0) + f(0,1) + \right.$$

$$\left. +4\left[f\left(\frac{1}{2}, 0\right) + f\left(\frac{1}{2}, \frac{1}{2}\right)\right] + f(1,0) + f(1,0)\right] + R(f)$$

Thus

Theorem 2. *If* $f \in B_{12}(0,0)$, *then:*

$$\iint_T f(x,y)dxdy = \frac{1}{12} f(0,0) + \frac{1}{12} f(0,1) + \frac{1}{3} f\left(\frac{1}{2}, 0\right) +$$

$$+\frac{1}{3} f\left(\frac{1}{2}, \frac{1}{2}\right) + \frac{1}{6} f(1,0) + R(f) \tag{6}$$

where

$$R(f) = -\frac{1}{30} f^{(3,0)}(\xi_1, 0) - \frac{1}{16} f^{(2,1)}(\xi_2, 0) - \frac{1}{18} f^{(0,3)}(0, \eta_1) - \frac{13}{240} f^{(1,2)})(\xi_3, \eta_3)$$

with $\xi_1, \xi_2, \eta_1 \in [0,1]$ *and* $(\xi_3, \eta_3) \in T_1$.

Another way to obtain the cubature formulas is to start from an approximation formula on T_1.

Let B_1 be the Birkhoff's operator:

$$(B_1 f)(x,y) = f(1-y, y) + (x+y-1) f^{(1,0)}(0, y)$$

which generates the approximation formula

$$f = B_1 f + Rf.$$

After integration on T it is obtained:

$$\iint\limits_T f(x,y)dxdy = \int_0^1 (1-y)f(1-y,y)dy - \frac{1}{2}\int_0^1 (y-1)^2 f^{(1,0)}(0,y)dy.$$

Applying to each integrals the Simpson's quadrature, it is obtained:

$$\int_0^1 (1-y)f(1-y,y)dy \simeq \frac{1}{6}f(1,0) + \frac{1}{3}f\left(\frac{1}{2},\frac{1}{2}\right)$$

$$\int_0^1 (1-y)^2 f^{(1,0)}(0,y)dy \simeq \frac{1}{6}\left[f^{(1,0)}(0,0) + f^{(1,0)}\left(0,\frac{1}{2}\right)\right].$$

It follows:

Theorem 3. *If $f \in B_{11}(0,0)$, then:*

$$\iint\limits_T f(x,y)dxdy = \frac{1}{6}\left[f(1,0) + 2f\left(\frac{1}{2},\frac{1}{2}\right)\right] -$$

$$-\frac{1}{12}\left[f^{(1,0)}(1,0) + f^{(1,0)}\left(0,\frac{1}{2}\right)\right] + R(f) \tag{7}$$

where:

$$R(f) = -\frac{1}{12}f^{(2,0)}(\xi,0), \quad \xi \in [0,1].$$

Starting from Lagrange's operator:

$$(L_1 f)(x,y) = \frac{1-x-y}{1-y}f(0,y) + \frac{x}{1-y}f(1-y,y)$$

and Hermite operator

$$(H_2 f)(x,y) = \frac{(1-x-y)(1-x+y)}{(1-x)^2}f(x,0) +$$

$$+\frac{y(1-x-y)}{1-x}f^{(0,1)}(x,0) + \frac{y^2}{(1-x)^2}f(x,1-x)$$

one obtains an interpolation formula

$$f = L_1 H_2 f + R_{12} f$$

where

$$(L_1 H_2 f)(x,y) = (1+y)(1-x-y)f(0,0) + x(1-x-y)f^{(0,1)}(0,0) +$$

$$+\frac{y^2(1-x-y)}{1-y}f(0,1) + \frac{x}{1-y}f(1-y,y).$$

After integration on T, one obtain:

$$\iint_T f(x,y)dxdy = f(0,0)\int_0^1\int_0^{1-x}(1+y)(1-x-y)dxdy+$$

$$+f^{(0,1)}(0,0)\int_0^1\int_0^{1-x}x(1-x-y)dxdy + f(0,1)\int_0^1\int_0^{1-x}\frac{y^2(1-x-y)}{1-y}dxdy+$$

$$+\int_0^1\int_0^{1-x}\frac{x}{1-y}f(1-y,y)dxdy =$$

$$=\frac{5}{24}f(0,0)+\frac{1}{24}f^{(0,1)}(0,0)+\frac{1}{24}f(0,1)+\int_0^1\int_0^{1-x}\frac{x}{1-y}f(1-y,y)dxdy.$$

In order to compute the integral $\int_0^1\int_0^{1-x}\frac{x}{1-y}f(1-y,y)dxdy$, we use the following cubature formulas:

$$\iint_T f(x,y)dxdy = \frac{1}{6}\left[g\left(\frac{1}{2},0\right)+g\left(0,\frac{1}{2}\right)+g\left(\frac{1}{2},\frac{1}{2}\right)\right]+R(f)$$

and we obtain

$$\int_0^1\int_0^1\frac{x}{1-y}f(1-y,y)dxdy \simeq \frac{1}{12}f(1,0)+\frac{1}{6}f\left(\frac{1}{2},\frac{1}{2}\right).$$

Thus, we arrive at the following result:

Theorem 4. *If $f\in B_{11}(0,0)$, then:*

$$\iint_T f(x,y)dxdy = \frac{5}{24}f(0,0)+\frac{1}{24}f^{(0,1)}(0,0)+\frac{1}{12}f(0,1)+\frac{1}{6}f\left(\frac{1}{2},\frac{1}{2}\right)+R(f) \quad (8)$$

where:

$$R(f) = -\frac{1}{24}f^{(2,0)}(\xi,0),\quad \xi\in[0,1].$$

References

[1] Barnhill, R.E., Birkhoff, G., Gordon, W.J., *Smooth interpolation in triangles*, J. Approx. Theory, 8(1973), 114-128.

[2] Barnhill, R.E., Mansfield, L., *Error bounds for smooth interpolation in triangles*, J. Approx. Theory, 11(1974), 306-318.

[3] Böhmer, K., Coman, Gh., *On some approximation schemes on triangles*, Mathematica (Cluj), 24(45), 1980, 231-235.

[4] Coman, Gh., *Analiză numerică*, Ed. Libris, Cluj-Napoca, 1995.

[5] Cools, R., Lyness, J.N., *A survey of numerical cubature over triangles*, Mathematics and Computer Science Divisio, Argonne National Laboratory, Argone III, 1994.

[6] Cools, R., *Monomial cubature rules since "Stroud" a compilation - part 2*, J. Comput. Appl. Math. 112(1999), 21-27.

[7] Cools, R., Mysovskikh, I.P., Schmid, H.J., *Cubature formulae and orthogonal polynomials*, Elsevier Preprint, January 2000.

[8] Stancu, D.D., *The remainder of certain linear approximation formulas in two variables*, SIAM J. Numer. Anal., Ser.B, 1(1964), 137-163.
[9] Stroud, A.H., *Approximate Calculation of Multiple Integrals*, Prentice-Hall, Englewood Cliffs, New York, 1971.

Faculty of Mathematics and Computer Science,
Babeş-Bolyai University, 3400 Cluj-Napoca, Romania
E-mail address: milenas73@yahoo.com

Received: 11.10.2001