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A UNIVALENCE CONDITION

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Abstract. In this paper we obtain a sufficient condition for univalence concerning holomorphic mappings of the unit ball in the space of n-complex variables.

1. Introduction

Let $\mathbf{C}^n$ be the space of n-complex variables $z = (z_1, ..., z_n)$ with the Euclidean inner product $\langle z.w\rangle = \sum_{k=1}^{n} z_k \bar{w}_k$ and norm $\|z\| = \langle z, z\rangle^{\frac{1}{2}}$.

Let B^n denote the open unit ball in $\mathbf{C}^n$,i.e. $B^n = \{z \in \mathbf{C}^n : \|z\| < 1\}$.We denote by $\mathcal{L}(\mathbf{C}^n)$ the space of continuous linear operators from $\mathbf{C}^n$ into $\mathbf{C}^n$,i.e $.n \times n$ complex matrices $A = (A_{jk})$ with the standard operator norm

$$\|A\| = \sup\{\|Az\| : \|z\| < 1\}, \qquad A \in \mathcal{L}(\mathbf{C}^n)$$

$I = (I_{jk})$ denotes the identity in $\mathcal{L}(\mathbf{C}^n)$.

Let $H(B^n)$ be the class of holomorphic mappings

$$f(z) = (f_1(z), ..., f_n(z)), \qquad z \in B^n$$

from B^n into $\mathbf{C}^n$. We say that $f \in H(B^n)$ is *locally biholomorphic* in B^n if f has a local holomorphic inverse at each point in B^n or equivalently, if the derivative

$$Df(z) = \left(\frac{\partial f_k(z)}{\partial z_j}\right)_{1 \le j,k \le n}$$

is nonsingular at each point $z \in B^n$.

A mapping $v \in H(B^n)$ is called a *Schwarz function* if $\|v(z)\| \le \|z\|$, for all $z \in B^n$.

If $f, g \in H(B^n)$ then f is *subordinate* to g $(f \prec g)$ in B^n if there exists a Schwarz function v such that $f(z) = g(v(z))$, $z \in B^n$.

A function $L : B^n \times [0,\infty) \to \mathbf{C}^n$ is a *subordination chain* if $L(\cdot,t)$ is holomorphic and univalent in $B^n, L(0,t) = 0$, for all $t \in [0,\infty)$ and $L(z,s) \prec L(z,t)$, whenever $0 \leq s \leq t < \infty$.

The subordination chain $L : B^n \times [0,\infty) \to \mathbf{C}^n$ is a *normalized* subordination chain if $DL(0,t) = e^t I$, for $t \in [0,\infty)$.

A basic result in the theory of n-complex variables subordination chains is due to J. A. Pfaltzgraff.

Theorem 1. [5] *Let $L(z,t) = e^t z + ...$ be a function from $B^n \times [0,\infty)$ into $\mathbf{C}^n$ such that:*

(i) $L(\cdot,t) \in H(B^n)$, for all $t \in [0,\infty)$

(ii) $L(z,t)$ is a locally absolutely continuous function of t, locally uniformly with respect to $z \in B^n$.

Let $h(z,t)$ be a function from $B^n \times [0,\infty)$ into $\mathbf{C}^n$ which satisfies the following conditions:

(iii) $h(\cdot,t) \in H(B^n), h(0,t) = 0, Dh(0,t) = I$ and $\operatorname{Re}\langle h(z,t), z\rangle \geq 0$, for all $t \in [0,\infty)$ and $z \in B^n$.

(iv) For each $T > 0$ and $r \in (0,1)$ there is a number $K = K(r,T)$ such that $\|h(z,t)\| \leq K(r,T)$, when $\|z\| \leq r$ and $t \in [0,T]$.

(v) For each $z \in B^n, h(z,\cdot)$ is a measurable function on $[0,\infty)$.

Suppose $h(z,t)$ satisfies

$$\frac{\partial L(z,t)}{\partial t} = DL(z,t)\,h(z,t)\,,\ a.e\ \ t \in [0,\infty),\ for\ all\ z \in B^n \tag{1}$$

Further, suppose there is a sequence $(t_m)_{m\geq 0}, t_m > 0$ increasing to ∞ such that

$$\lim_{m\to\infty} e^{-t_m} L(z,t_m) = F(z) \tag{2}$$

locally uniformly in B^n.

Then for each $t \in [0,\infty), L(\cdot,t)$ is univalent in B^n.

P. Curt obtained a version of Theorem 1 for subordination chains which are not normalized .

Theorem 2. [2] *Let $L(z,t) = a_1(t)z + ..., a_1(t) \neq 0$ be a function from $B^n \times [0,\infty)$ into $\mathbf{C}^n$ such that:*

(i) $L(\cdot,t) \in H(B^n)$ for all $t \in [0,\infty)$

(ii) $L(z,t)$ *is a locally absolutely continuous function of* t*, locally uniformly with respect to* $z \in B^n$

(iii) $a_1(t) \in C^1[0,\infty)$ *and* $\lim\limits_{t\to\infty} |a_1(t)| = \infty$.

Let $h(z,t)$ *be a function from* $B^n \times [0,\infty)$ *into* $\mathbf{C}^n$ *which satisfies the following conditions:*

(iv) $h(\cdot,t) \in H(B^n)$, $h(0,t) = 0$ *and* $\operatorname{Re}\langle h(z,t), z\rangle \geq 0$, *for all* $t \in [0,\infty)$ *and* $z \in B^n$

(v) For each $z \in B^n, h(z,\cdot)$ *is a measurable function on* $[0,\infty)$

(vi)For each $T > 0$ *and* $r \in (0,1)$, *there exists a number* $K = K(r,T)$ *such that* $\|h(z,t)\| \leq K(r,T)$, *when* $\|z\| \leq r$ *and* $t \in [0,T]$.

Suppose $h(z,t)$ *satisfies*

$$\frac{\partial L(z,t)}{\partial t} = DL(z,t)\, h(z,t)\,, a.e.\, t \in [0,\infty),\ \text{for all } z \in \mathbf{B}^n \tag{3}$$

Further suppose there is a sequence $(t_m)_{m\geq 0}, t_m > 0$ *increasing to* ∞ *such that*

$$\lim_{m\to\infty} \frac{L(z,t_m)}{a_1(t_m)} = F(z) \tag{4}$$

locally uniformly in B^n.

Then for each $t \in [0,\infty), L(\cdot,t)$ *is univalent in* B^n.

2. **Univalence conditions**

By using Theorem 2, we obtain an univalence condition which generalize some n-dimensional univalence criteria [2], [3], [5].

Theorem 3. *Let* $f : B^n \to \mathbf{C}^n$ *be a locally biholomorphic function in* $B^n, f(0) = 0, Df(0) = I$ *and let* $a : [0,\infty) \to \mathbf{C}$ *be a function which satisfies the conditions:*

(i) $a \in C^1[0,\infty), a(0) = 1, a(t) \neq 0$, *for all* $t \in [0,\infty)$

(ii) $\lim\limits_{t\to\infty} |a(t)| = \infty$

(iii) $\operatorname{Re} \frac{a'(t)}{a(t)} > 0$, *for all* $t \in [0,\infty)$.

If

$$\max_{\|z\|=e^{-t}} \left\| (a(t) - \|z\|)(Df(z))^{-1} D^2 f(z)(z,\cdot) + \frac{a(t) - a'(t)}{2} I \right\| < \\ < \frac{|a(t) + a'(t)|}{2} \tag{5}$$

for all $t \in [0,\infty)$*, then* f *is an univalent function in* B^n.

Remark

The second derivative of a function $f \in H(B^n)$ is a symmetric bilinear operator $D^2 f(z)(\cdot,\cdot)$ on $\mathbf{C}^n \times \mathbf{C}^n$ and $D^2 f(z)(w,\cdot)$ is the linear operator obtained by restricting $D^2 f(z)$ to $\{w\} \times \mathbf{C}^n$. The linear operator $D^2 f(z)(z,\cdot)$ has the matrix representation

$$D^2 f(z)(z,\cdot) = \left(\sum_{m=1}^{n} \frac{\partial^2 f_k(z)}{\partial z_j \partial z_m} z_m \right)_{1 \le j,k \le n}$$

Proof. We define

$$L(z,t) = f\left(e^{-t}z\right) + \left(a(t)e^{t} - 1\right) e^{-t} Df\left(e^{-t}z\right)(z), \quad t \in [0,\infty), z \in B^n \tag{6}$$

We wish to show that $L(z,t)$ satisfies the conditions of Theorem 2 and hence $L(\cdot,t)$ is univalent in B^n, for all $t \in [0,\infty)$. Since $f(z) = L(z,0)$ we obtain that f is an univalent function in B^n.

It is easy to check that $a_1(t) = a(t)$ and hence $a_1(t) \neq 0, \lim t \to \infty |a_1(t)| = \infty$ and $a_1 \in C^1[0,\infty)$.

We have $L(z,t) = a_1(t)z + (holomorphicterm)$. Thus $\lim\limits_{t\to\infty} \frac{L(z,t)}{a_1(t)} = z$, locally uniform with respect to $B^n$and hence (4) holds with $F(z) = z$. Obviously $L(z,t)$ satisfies the absolute continuity requirements of Theorem2.

Straightforward calculations show that

$$DL(z,t) = \frac{a(t) + a'(t)}{2} Df\left(e^{-t}z\right)\left[I - E(z,t)\right], \tag{7}$$

where, for each fixed $(z,t) \in B^n \times [0,\infty), E(z,t)$ is the linear operator defined by

$$E(z,t) = -\frac{a(t) - a'(t)}{a(t) + a'(t)} I -$$

$$-2\frac{a(t) - e^{-t}}{a(t) + a'(t)} \left(Df\left(e^{-t}z\right)\right)^{-1} D^2 f\left(e^{-t}z\right)\left(e^{-t}z,\cdot\right) \tag{8}$$

For $t = 0$, we have

$$I - E(z,0) = \frac{2}{1 + a'(0)} I, \quad \text{for all } z \in B^n \tag{9}$$

Since $1 + a'(0) \neq 0$, we obtain that $I - E(z,0)$ is an invertible operator.

For $t > 0, E(\cdot,t) : \overline{B^n} \to \mathcal{L}(\mathbf{C}^n, \mathbf{C}^n)$ is holomorphic and from the weak maximum modulus theorem [4] it follows that $\|E(z,t)\|$ can have no maximum in B^n unless $\|E(z,t)\|$ is of constant value throughout B^n. If $z = 0$ and $t > 0$ we have

$$\|E(0,t)\| = \left| \frac{a(t) - a'(t)}{a(t) + a'(t)} \right| < 1.$$

We also have

$$\|E(z,t)\| \leq \max_{\|w\|=1} \|E(w,t)\|$$

If we let $u = e^{-t}w$ with $\|w\| = 1$, then $\|u\| = e^{-t}$ and by using (5) we obtain

$$\|E(z,t)\| \leq \max_{\|w\|=1} \|E(w,t)\| =$$

$$= \max_{\|u\|=e^{-t}} \left\| \frac{2(a(t) - \|u\|)}{a(t) + a'(t)} (Df(u))^{-1} D^2 f(u)(u,\cdot) + \frac{a(t) - a'(t)}{a(t) + a'(t)} I \right\| < 1.$$

Since $\|E(z,t)\| < 1$ for all $z \in B^n$ and $t > 0$, it follows $I - E(z,t)$ is an invertible operator,too.

Further calculations show that

$$\frac{\partial L(z,t)}{\partial t} = \frac{a(t) + a'(t)}{2} Df\left(e^{-t}z\right)[I - E(z,t)](z) =$$

$$DL(z,t) = [I - E(z,t)]^{-1}[I + E(z,t)](z).$$

Hence $L(z,t)$ satisfies the differential equation (3), for all $z \in B^n$ and $t \in [0,\infty)$, where

$$h(z,t) = [I - E(z,t)]^{-1}[I + E(z,t)](z) \tag{10}$$

It remains to show that $h(z,t)$ satisfies the conditions $(iv), (v)$ and (vi) of Theorem 2. Clearly $h(z,t)$ satisfies the holomorphy and measurability requirements and $h(0,t) = 0$.

Since

$$\|h(z,t) - z\| = \|E(z,t)(h(z,t) + z)\| \leq \|E(z,t)\| \cdot \|h(z,t) + z\| < \|h(z,t) + z\|$$

We have $\langle \operatorname{Re} h(z,t), z \rangle \geq 0$, for all $(z,t) \in B^n \times [0,\infty)$.

By using the inequality

$$\left\|[I - E(z,t)]^{-1}\right\| \leq [1 - \|E(z,t)\|]^{-1}$$

we obtain

$$\|h(z,t)\| \leq \frac{1 + \|E(z,t)\|}{1 - \|E(z,t)\|} \|z\|.$$

The conditions of Theorem 2 being satisfied it follows that the functions $L(z,t), t \geq 0$ are univalent in B^n. In particular $f(z) = L(z,0)$ is univalent in B^n.

Remarks

1) If $a(t) = e^t, t \in [0,\infty)$, then Theorem 3 becomes the n-dimensional version of Becker's univalence criterion [4].

2) For $a(t) = \frac{e^t + ce^{-t}}{1+c}, t \geq 0, c \in \mathbf{C} \setminus \{-1\}, |c| \leq 1$, Theorem 3 becomes the n-dimensional version of Ahlfors and Becker's univalence criterion [2].

3) If $a(t) = \frac{e^{(\alpha-1)t} + ce^{-t}}{1+c}, t \geq 0, c \in \mathbf{C} \setminus \{-1\}, |c| \leq 1$ and $\alpha \in \mathbf{R}$ with $\alpha \geq 2$, we obtain the generalization of Ahlfors and Becker's n-dimensional criterion of univalence due to P. Curt [3].

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