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ON UNIFORMLY CONVEX MAPPINGS OF A BANACH SPACE INTO THE COMPLEX PLANE

DORINA RĂDUCANU

Abstract. Let E be a complex Banach space and let E be the unit ball in E, i.e. $B = \{x \in E : \|x\| < 1\}$. We introduce a new class of holomorphic functions in B and we obtain a few results concerning this new class.

1. **Introduction**

Let E^*be the dual space of E. For any $A \in E^*$we consider $\chi(A) = \{x \in E : A(x) \neq 0\}$ and $\gamma(A) = E \backslash \chi(A)$. If $A \neq 0$ then $\chi(A)$ is dense in E and $\chi(A) \cap \hat{B}$ is dense $\hat{B}$, where $\hat{B} = \{x \in E : \|x\| = 1\}$.

Let $H(B)$ be the family of all functions $f : B \to \mathbf{C}, f(0) = 0$ that are holomorphic in B, i.e. have the Fréchet derivative $f'(x)$ in each point $x \in B$. If $f \in H(B)$, then in some neighbourhood V of the origin, $f(x) = \sum\limits_{m=1}^{\infty} P_{m,f}(x)$, where the series is uniformly convergent on V and

$P_{m,f} : E \to \mathbf{C}$ are continuous and homogeneous polynomials of degree m.

Let $U = \{z \in \mathbf{C} : |z| < 1\}$. Denote by CV the family of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

that are convex in the unit disk U.

Goodman [1] defined the following subclass of CV.

Definition. A function f is called uniformly convex in U if f is in CV and has the property that for every circular arc γ contained in U, with center ζ also in U, the arc $f(\gamma)$ is a convex arc.

Goodman gave a two-variable analytic characterization of this class, denoted by UVC.

Theorem 1. *A function of the form* (1) *is in* UCV *if and only if*

$$\operatorname{Re}\left\{1+(z-\zeta)\frac{f''(z)}{f'(z)}\right\}\geq 0 \quad , \quad (z,\zeta)\in U\times U. \tag{2}$$

Also, Goodman proved that the best known bounds on the coefficients for the family UVC are $|a_n|\leq \frac{1}{n}, n\geq 2$.

Ma and Minda [3] and Ronning [4] independently found a more applicable one-variable characterization for UVC.

Theorem 2. *A function* f *of the form* (1) *is in* UVC *if and only if*

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}\geq \left|\frac{zf''(z)}{f'(z)}\right| \quad , \quad z\in U. \tag{3}$$

2. **The class** UCV_A

Let $A\in E^*, A\neq 0$. For any $f\in H(B)$ of the form

$$f(x)=A(x)+\sum_{n=2}^{\infty}P_{n,f}(x) \quad , \quad x\in B \tag{4}$$

and for any $a\in\chi(A)\cap\hat{B}$ we set

$$f_a(z)=\frac{f(za)}{A(a)} \quad , \quad z\in U. \tag{5}$$

Obviously

$$f_a(z)=z+\sum_{n=2}^{\infty}\frac{P_{n,f}(a)}{A(a)}z^n \quad , \quad z\in U. \tag{6}$$

Moreover, it is easy to check that

$$f_a^{(n)}(z)=\frac{f^{(n)}(za)(a,...,a)}{A(a)} \quad , \quad n\in\mathbf{N}, z\in U. \tag{7}$$

We denote by UCV_A the family of all functions $f\in H(B)$ of the form (4) such that, for any $a\in\chi(A)\cap\hat{B}$ the function f_a belongs to the class UCV.

By using the properties of the functions in UCV, we obtain a few results concerning the family UCV_A.

Theorem 3. *If* $f\in UCV_A$ *and* $a\in\hat{B}$, *then*

$$|P_{n,f}(a)|\leq\frac{1}{n}|A(a)| \quad , \quad n\geq 2 \tag{8}$$

Proof. Suppose that $f \in UCV_A$. If $a \in \chi(A) \cap \hat{B}$, then $f_a \in UCV$ and hence we get (9). If $a \in \gamma(A) \cap \hat{B}$, evidently $a = \lim_{m\to\infty} a_m$, where $a_m \in X(A), m \in \mathbf{N}$. There exists $r_m \in \mathbf{R}_+$ such that $\frac{a_m}{r_m} \in \hat{B}$. Clearly $(r_m)_{m\geq 0}$ is bounded for the origin is an interior point of B. Since $\frac{a_m}{r_m} \in \chi(A) \cap \hat{B}, m \in \mathbf{N}$, by the first part of the proof we have

$$\left|P_{n,f}\left(\frac{a_m}{r_m}\right)\right| \leq \frac{1}{n}\left|A\left(\frac{a_m}{r_m}\right)\right| \quad , \quad m \in \mathbf{N}.$$

Hence

$$|P_{n,f}(a_m)| \leq \frac{r_m^{n-1}}{n}|A(a_m)| \quad , \quad m \in \mathbf{N}.$$

By taking the limit with $m \to \infty$, we obtain $P_{n,f}(a) = 0$.

Corollary 1. *All $f \in UCV_A$ vanish on $\gamma(A) \cap B$.*

Corollary 2. *If $f \in UCV_A$, then*

$$\|P_{n,f}\| \leq \frac{1}{n}\|A\| \quad , \quad n \geq 2$$

The following theorems provide necessary and sufficient conditions for functions in $H(B)$ to belong to the class UCV_A.

Theorem 4. *Let $f \in UCV_A$ and $f'(x) \neq 0$, for all $x \in B$. Then*

$$\operatorname{Re}\left\{1 + \frac{f''(x)(x,x)}{f'(x)}\right\} \geq \left|\frac{f''(x)(x,x)}{f'(x)}\right| \quad , \quad x \in \chi(A) \cap B. \tag{9}$$

Proof. Let $x \in \chi(A) \cap B, x \neq 0$. Then $a = \frac{x}{\|x\|} \in \chi(A) \cap \hat{B}$ and hence the function f_a belongs to the class UCV. From (3) we have

$$\operatorname{Re}\left\{1 + \frac{zf_a''(z)}{f_a'(z)}\right\} \geq \left|\frac{zf_a''(z)}{f_a'(z)}\right| \quad , \quad z \in U.$$

By using the equality

$$\frac{zf_a''(z)}{f_a'(z)} = \frac{f''(za)(za,za)}{f'(za)(za)} \quad , \quad z \in U$$

we obtain

$$\operatorname{Re}\left\{1 + \frac{f''(za)(za,za)}{f'(za)(za)}\right\} \geq \left|\frac{f''(za)(za,za)}{f'(za)(za)}\right| \quad , \quad z \in U.$$

By setting $z = \|x\|$, we get (9).

Theorem 5. *Let $f \in H(B), f'(0) = A$ and $f'(x) \neq 0$, for all $x \in B$. If*

$$\operatorname{Re}\left\{1 + \frac{f''(x)(x,x)}{f'(x)}\right\} \geq \left|\frac{f''(x)(x,x)}{f'(x)}\right| \quad , \quad x \in B \tag{10}$$

then $f \in UCV_A$.

Proof. Let $a \in \chi(A) \cap \hat{B}$. Then $f'_a(z) = f'(za)(a) \neq 0, z \in U \setminus \{0\}$ and

$$\frac{zf''_a(z)}{f'_a(z)} = \frac{f''(za)(za,za)}{f'(za)} \quad , \quad z \in U.$$

From (10), we obtain $f_a \in UCV$, for all $a \in \chi(A) \cap \hat{B}$. Hence $f \in UCV_A$.

References

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Department of Mathematics, Faculty of Sciences,
"Transilvania" University, Iuliu Maniu, 50, 2200 Braşov, Romania

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