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ON A QUOTIENT CATEGORY

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Abstract. We construct a suitable quotient category, in order to give natural interpretations of the notions almost projective module and almost injective module.

Introduction

The study of finite rank torsion free abelian groups using quasi-notions (see [2]) imposed in module theory the notions as "almost projective module", "almost injective module" and "almost flat module" ([1], [6], [9], [10]). In [11], E. Walker give a natural setting for the study of quasi-homomorphisms of abelian group, constructing the quotient category of the category of the abelian groups modulo the Serre class of all bounded groups, and he shows, that this quotient category is equivalent to the category constructed by Reid in [7], in order to give a categorial interpretation of the B. Jónson's quasi-decomposition theorem (see [2, Corollary 7.9]).

In this paper we show, using a analogous construction to the E. Walker's one, that the mentioned "almost-notions" have natural interpretations in a suitable quotient category.

1. **The basic construction**

Let $\mathcal{A}$ be an additive category and $S \subseteq \mathbb{N}^*$ be a multiplicative system such that $1 \in S$. We consider the class of all homomorphisms of the form $n_A = n1_A$, with $A \in \mathcal{A}$ and $n \in S$, and we denote it by Σ. Then, for each $A \in \mathcal{A}$, the class of all homomorphisms belonging to Σ having the domain or the codomain A are sets, being subclasses of the set $\mathrm{End}_{\mathcal{A}}(A)$. Thus the left, respectively right, fractional category of $\mathcal{A}$ with respect Σ is defined [5, chap. 1, 14.6]. Moreover, it is straightforward

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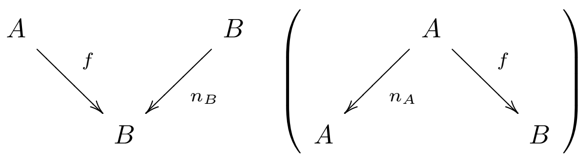
to check that Σ is a bicalculable multiplicative system of homomorphisms in $\mathcal{A}$ in the sense of [5, chap. 1, 1.14] so, the notions of left fractional category, the right fractional category and the category of additive fraction of $\mathcal{A}$ with respect Σ coincide [5, chap. 1, 14.5 and chap. 4, 7.5]. We shall denote by $\mathcal{A}[\Sigma^{-1}]$ this category, and by $\mathbf{q}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}[\Sigma^{-1}]$ (or simply $\mathbf{q}$ if there is no danger of confusion) the canonical functor. Recall that, this functor makes invertible the homomorphisms belonging to Σ, and satisfies the following universal property: for every category $\mathcal{B}$ and for every functor $F : \mathcal{A} \to \mathcal{B}$ making invertible all homomorphisms of Σ, there is a unique functor $\bar{F} : \mathcal{A}[\Sigma^{-1}] \to \mathcal{B}$ such that $\bar{F}\mathbf{q} = F$. Note that we may consider, the objects of $\mathcal{A}[\Sigma^{-1}]$ are the same as the objects of $\mathcal{A}$, in which case $\mathbf{q}(A) = A$ for all $A \in \mathcal{A}$, and

$$\mathrm{Hom}_{\mathcal{A}[\Sigma^{-1}]}(A,B) = \{\mathbf{q}(n_B)^{-1}\mathbf{q}(f)\mathbf{q}(m_A)^{-1} = \mathbf{q}(nm_B)^{-1}\mathbf{q}(f) = \mathbf{q}(f)\mathbf{q}(nm_A)^{-1} \mid f \in \mathrm{Hom}_{\mathcal{A}}(A,B) \text{ and } n, m \in S\}.$$

Since the category $\mathcal{A}[\Sigma^{-1}]$ may be seen as the category of left (right) fractions, the homomorphism $\mathbf{q}(n_B)^{-1}\mathbf{q}(f) : A \to B$ in this category may be visualized as a diagram of homomorphisms in $\mathcal{A}$:



Keeping in the mind that $f = 1_B^{-1}f$, we shall denote sometimes the above homomorphism by $n^{-1}f$ or, how we shall see, by $\frac{1}{n} \otimes f$.

Note that the functor $\mathbf{q}$ is left and right exact (that is, it commutes with finite limits and colimits), hence if $\mathcal{A}$ is finitely complete or finitely cocomplete then $\mathcal{A}[\Sigma^{-1}]$ has the same property [5, chap. 1, 14.5]. Moreover, if $\mathcal{A}$ is an abelian category, then $\mathcal{A}[\Sigma^{-1}]$ is abelian too [5, chap. 4, 7.6].

Let $\mathcal{A}$ be a full subcategory of an additive category $\mathcal{B}$. Let $\mathcal{A}'$ be the full subcategory of $\mathcal{B}[\Sigma^{-1}]$ consisting of those objects $\mathbf{q}(A)$, with $A \in \mathcal{A}$. The inclusion functor $\mathbf{i} : \mathcal{A} \to \mathcal{B}$ induce a fully faithful functor $\bar{\mathbf{i}} : \mathcal{A}[\Sigma^{-1}] \to \mathcal{B}[\Sigma^{-1}]$ which factors through $\mathcal{A}'$, the induced functor being representative, hence it is an equivalence.

In the sequel, we assume that *the category $\mathcal{A}$ is a locally small (that is, the subobjects of an object form a set) abelian category.* A full subcategory $\mathcal{T}$ of the category $\mathcal{A}$ is called *thick* or *Serre class* if, for every exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$, $B \in \mathcal{T}$ if and only if $A \in \mathcal{T}$ and $C \in \mathcal{T}$. Then, the class of homomorphisms in $\mathcal{A}$

$$\Sigma_{\mathcal{T}} = \{f| \ \mathrm{Ker}(f) \in \mathcal{T}, \ \mathrm{Coker}(f) \in \mathcal{T}\}$$

is a bicalculable multiplicative system and the category $\mathcal{A}[\Sigma^{-1}]$ is defined [5, chap. 4, 7.7 and 7.8], and it is denoted by $\mathcal{A}/\mathcal{T}$. In addition, the canonical functor carries all object from $\mathcal{T}$ into 0, and we can give the universal property with functors satisfying this condition [5, chap 4, exercise 7.3, f)] (see also [4, Section III]).

Remark 1.1 Using the construction of the fractional category of $\mathcal{A}$ with respect to $\Sigma_{\mathcal{T}}$, [5, Theorem 1.14.1], we observe that in this case the category $\mathcal{A}/\mathcal{T}$ is exactly the category constructed by Gabriel in [4]. We recall that in this construction the groups of homomorphisms in the quotient category $\mathcal{A}/\mathcal{T}$ which is induced by a Serre class $\mathcal{T}$ are the limit

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{T}}(A,B) = \varinjlim_{A/A' \in \mathcal{T}, B' \in \mathcal{T}} \mathrm{Hom}(A', B/B')$$

and the operations are canonical.

Returning to our case, we say that an object $A \in \mathcal{A}$ is *S-bounded* if there is $n \in S$ such that $n_A = 0$. If more precision is required, then the object A is called *bounded by $n \in S$*. It is straightforward to check that the class of all S-bounded objects of $\mathcal{A}$ forms a thick subcategory, the extremes of an extension being bounded by the same integer as the middle term, and this by the product of the integers which bound the extremes. We denote by $\mathcal{S}$ this subcategory. Clearly, the categories $\mathcal{A}[\Sigma^{-1}]$ and $\mathcal{A}/\mathcal{S}$ are isomorphic.

Lemma 1.2. *Let $\mathcal{A}$ be an abelian category, $1 \in S \subseteq \mathbb{N}^*$ and $\Sigma = \{n_A | A \in \mathcal{A}, n \in S\}$. Then the following hold for the category $\mathcal{A}[\Sigma^{-1}]$:*

a) *The homomorphism $\mathbf{q}(f)$ is a monomorphism in $\mathcal{A}[\Sigma^{-1}]$ if and only if* $\ker f$ *is S-bounded, for every homomorphism f in $\mathcal{A}$. The homomorphism $n^{-1}f$ is a monomorphism in $\mathcal{A}[\Sigma^{-1}]$ if and only if $\mathbf{q}(f)$ is a monomorphism in $\mathcal{A}[\Sigma^{-1}]$.*

b) *The homomorphism* $\mathbf{q}(f)$ *is an epimorphism in* $\mathcal{A}[\Sigma^{-1}]$ *if and only if* $\operatorname{coker} f$ *is* S*-bounded, for every homomorphism* f *in* $\mathcal{A}$*. The homomorphism* $n^{-1}f$ *is an epimorphism in* $\mathcal{A}[\Sigma^{-1}]$ *if and only if* $\mathbf{q}(f)$ *is an epimorphism in* $\mathcal{A}[\Sigma^{-1}]$*.*

c) *The homomorphism* $\mathbf{q}(f)$ *is an isomorphism in* $\mathcal{A}[\Sigma^{-1}]$ *if and only if* $\ker f$ *and* $\operatorname{coker} f$ *are* S*-bounded, for every homomorphism* f *in* $\mathcal{A}$*. The homomorphism* $n^{-1}f$ *is an isomorphism in* $\mathcal{A}[\Sigma^{-1}]$ *if and only if* $\mathbf{q}(f)$ *is an isomorphism in* $\mathcal{A}[\Sigma^{-1}]$*.*

Proof. The sentences relative to f are easy consequences of exactness of $\mathbf{q}$. But n_A^{-1} are isomorphisms in $\mathcal{A}[\Sigma^{-1}]$, for all $A \in \mathcal{A}$, and this completes the proof.

As is [11] we consider the category $\mathbb{Z}[S^{-1}]\mathcal{A}$, whose objects are the same as the objects of $\mathcal{A}$, and the homomorphisms sets are

$$\operatorname{Hom}_{\mathbb{Z}[S^{-1}]\mathcal{A}}(A, B) \cong \mathbb{Z}[S^{-1}] \otimes \operatorname{Hom}_{\mathcal{A}}(A, B),$$

for all $A, B \in \mathcal{A}$. The proof of the following results is inspired by the E. Walker's proof of [11, Theorem 3.1].

Proposition 1.3. *The categories* $\mathcal{A}[\Sigma^{-1}]$ *and* $\mathbb{Z}[S^{-1}]\mathcal{A}$ *are isomorphic.*

Proof. We shall view the quotient category $\mathcal{A}/\mathcal{S}$ as in Remark 1.1. If $n \in S$ and $A \in \mathcal{A}$, then we will denote $A[n] = \operatorname{Ker}(n_A)$ and $nA = \operatorname{Im}(n_A)$. If $n'_A : A \to nA$ is the canonical epimorphism n_A and $i_n = i_n^A : A[n] \to A$ is the canonical monomorphism, we have the exact sequence

$$0 \to A[n] \overset{i_n^A}{\to} A \overset{n'_A}{\to} nA \to 0.$$

Moreover, denote by $\alpha_n = \alpha_n^A : nA \to A/A[n]$ the canonical isomorphism and by $p_n = p_n^A = \alpha_n^A n'_A$ the canonical projection.

Let $f : A \to C$ be a homomorphism in $\mathcal{A}$ and $n \in S$. Then the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A[n] & \xrightarrow{i_n^A} & A & \xrightarrow{p_n^A} & A/A[n] & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle f} & & & & \\ 0 & \longrightarrow & C[n] & \xrightarrow{i_n^C} & C & \xrightarrow{p_n^C} & C/C[n] & \longrightarrow & 0 \end{array}$$

and the equalities $n_C f i_n^A = f n_A i_n^A = 0$ imply that there exists a homomorphism $f_n : A/A[n] \to C/C[n]$ such that $f_n p_A^n = p_C^n f$. Then $f_n \alpha_n : nA \to C/C[n]$. The objects A/nA and $C[n]$ are in the class $\mathcal{S}$, so $f_n\alpha_n$ represents a homomorphism $\overline{f_n\alpha_n} \in \operatorname{Hom}_{\mathcal{A}/\mathcal{B}}(A, C)$.

We consider the additive functor $F : \mathbb{Z}[S^{-1}]\mathcal{A} \to \mathcal{A}/\mathcal{B}$ with $F(A) = A$ and $F(\frac{1}{n}f) = \overline{f_n\alpha_n}$ for every homomorphism $\frac{1}{n}f$ in $\mathbb{Z}[S^{-1}]\mathcal{A}$.

In order to construct the inverse of F, we define a functor $G : \mathcal{A}/\mathcal{B} \to \mathbb{Z}[S^{-1}]\mathcal{A}$, putting $G(A) = A$ for every object A, and for homomorphisms constructing the image by G in the following way: Let $\overline{g} \in \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(A, B)$ be a homomorphism which is represented by $g' : A' \to B/B'$ and $n \in S$ such that $n_{A/A'} = 0$ and $n_{B'} = 0$. Then we can suppose that $\overline{g}$ is represented by $g : nA \to B/B[n]$. We put $G(\overline{g}) = \frac{1}{n^2}((\alpha_n^B)^{-1}gn'_A)$.

We denote by g' the composition between the restriction of g to n^2A with the canonical projection $\pi : B/B[n] \to B/B[n^2]$ and we obtain

$$g'(\alpha_{n^2}^A)^{-1}p_{n^2}^B = \pi g n^2 = \pi n g n = \pi p_n^B(\alpha_n^B)^{-1}gn = p_{n^2}^B(\alpha_n^B)^{-1}gn.$$

It follows, using the universal property of cokernels,

$$((\alpha_n^B)^{-1}gn)_{n^2} = g'(\alpha_{n^2}^A)^{-1}.$$

Thus

$$F(G(\overline{g})) = \overline{(\alpha_n^B)^{-1}gn)_{n^2}\alpha_{n^2}} = \overline{g'} = \overline{g}.$$

If $\frac{1}{n}f \in \mathbb{Z}[S^{-1}]\,\mathrm{Hom}(A, B)$, then

$$G(F(\frac{1}{n}f)) = G(\overline{f_n\alpha_n}) = \frac{1}{n^2}((\alpha_n^B)^{-1}f_n\alpha_n^Bn)) = \frac{1}{n^2}nf = \frac{1}{n}f,$$

and the proof is complete.

Proposition 1.4. *The canonical functor* $\mathbf{q} : \mathcal{A} \to \mathbb{Z}[S^{-1}]\mathcal{A}$ *preserves the injective objects and the projective objects.*

Proof. Let I be an injective object in $\mathcal{A}$ and let $0 \to A \overset{\frac{1}{n}\alpha}{\to} B$ be a monomorphism in $\mathbb{Z}[S^{-1}]\mathcal{A}$. Then $\alpha : A \to B$ is a homomorphism in $\mathcal{A}$ for which exists $n \in S$ with $n\,\mathrm{Ker}(\alpha) = 0$. Because I is an injective object, we obtain the exact sequence of abelian groups

$$\mathrm{Hom}(B, I) \to \mathrm{Hom}(A, I) \to \mathrm{Hom}(\mathrm{Ker}(\alpha), I) \to 0.$$

Applying the tensor product with S, which is an exact functor, because the group $\mathbb{Z}[S^{-1}]$ is torsion free, we find the exact sequence

$$\mathbb{Z}[S^{-1}]\,\mathrm{Hom}(B, I) \to \mathbb{Z}[S^{-1}]\,\mathrm{Hom}(A, I) \to \mathbb{Z}[S^{-1}]\,\mathrm{Hom}(\mathrm{Ker}(\alpha), I) \to 0$$

in which $\mathbb{Z}[S^{-1}]\operatorname{Hom}(\operatorname{Ker}(\alpha), I) = 0$ because for every $f \in \operatorname{Hom}(\operatorname{Ker}(\alpha), I)$ we have $nf = fn_{\operatorname{Ker}(\alpha)} = 0$, hence $n\operatorname{Hom}(\operatorname{Ker}(\alpha), I) = 0$. It follows that $\mathbf{q}(I)$ is an injective object in $\mathbb{Z}[S^{-1}]\mathcal{A}$.

Dual it may be proved that S preserves the projective objects.

Corollary 1.5. *If $\mathcal{A}$ is an abelian category with enough injective objects, then:*

a) $\mathbb{Z}[S^{-1}]\mathcal{A}$ has enough injective objects,

b) for all $A, C \in \mathcal{A}$ and $n \in S$ there exists the canonical isomorphisms $\operatorname{Ext}^n_{\mathbb{Z}[S^{-1}]\mathcal{A}}(C, A) \cong \mathbb{Z}[S^{-1}] \otimes \operatorname{Ext}^n_{\mathcal{A}}(C, A)$.

Proof. a) We choose an injective resolution for A in $\mathcal{A}$:

$$0 \to A \to I \to I/A \to 0,$$

and Proposition 1.4 proves that

$$0 \to \mathbf{q}(A) \to \mathbf{q}(I) \to \mathbf{q}(I/A) \to 0$$

represents an injective resolution in $\mathbb{Z}[S^{-1}]\mathcal{A}$.

b) We proceed by induction, using an injective resolution as in a). For the first step of the induction, observe that $\mathbb{Z}[S^{-1}] \otimes \operatorname{Ext}^1_{\mathcal{A}}(C, A)$ and $\operatorname{Ext}^1_{\mathbb{Z}[S^{-1}]\mathcal{A}}(C, A)$ are isomorphic, being both the cokernels of the homomorphism $\mathbb{Z}[S^{-1}]\operatorname{Hom}_{\mathcal{A}}(C, I) \to \mathbb{Z}[S^{-1}]\operatorname{Hom}_{\mathcal{A}}(C, A)$. Furthermore, for every $1 \leq n \in S$ we have the isomorphisms

$$\operatorname{Ext}^n_{\mathbb{Z}[S^{-1}]\mathcal{A}}(C, I/A) \cong \operatorname{Ext}^{n+1}_{\mathbb{Z}[S^{-1}]\mathcal{A}}(C, A),$$

respectively

$$\operatorname{Ext}^n_{\mathcal{A}}(C, I/A) \cong \operatorname{Ext}^{n+1}_{\mathcal{A}}(C, A).$$

Corollary 1.6. *If $\mathcal{A}$ has enough projective objects, then $\mathbb{Z}[S^{-1}]\mathcal{A}$ has the same property.*

2. **Almost projective and almost injective objects**

Throughout of this section, $\mathcal{A}$ *will be a locally small abelian category with enough projective and enough injective objects.* An object P of $\mathcal{A}$ is called *S-almost projective (injective)*, if $\mathbf{q}(P)$ is an projective (respectively injective) object in $\mathbb{Z}[S^{-1}]A$.

Lemma 2.1. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Then there exists a unique functor $\bar{F} : \mathbb{Z}[S^{-1}]\mathcal{A} \to \mathbb{Z}[S^{-1}]\mathcal{B}$ such that $\bar{F}\mathbf{q}_{\mathcal{A}} = \mathbf{q}_{\mathcal{B}}F$*

Proof. The statement is a consequence of the universal property of a quotient category modulo a Serre class and of the fact that, F being additive, for every $n \in S$ and for every $A \in \mathcal{A}$ we have $F(n_A) = n_{F(A)}$.

If A is an object in $\mathcal{A}$, then we shall denote by

$$\mathbf{H}_A = \mathrm{Hom}(A, -) : \mathcal{A} \to \mathcal{A}b$$

the canonical covariant functor, and by

$$\bar{\mathbf{H}}_A : \mathbb{Z}[S^{-1}]\mathcal{A} \to \mathbb{Z}[S^{-1}]\mathcal{A}b$$

the functor, which is induced by $\mathbf{H}$.

We recall that an abelian group G is a S-torsion group if the order of every element $g \in G$ is in S and G is S-bounded if there exists $n \in S$ such that $nG = 0$. We record the following characterization of the almost projective objects:

Proposition 2.2. *For a $P \in \mathcal{A}$, the following conditions are equivalent:*

(i) *P is S-almost projective;*

(ii) *The group $\mathrm{Ext}^1_{\mathcal{A}}(P, A)$ is S-torsion, for all $A \in \mathcal{A}$;*

(iii) *The group $\mathrm{Ext}^1_{\mathcal{A}}(P, A)$ is S-bounded, for all $A \in \mathcal{A}$;*

(iv) *There is an integer $n = n(P) \in S$, such that $n\,\mathrm{Ext}^1_{\mathcal{A}}(P, A) = 0$ for all $A \in \mathcal{A}$;*

(v) *There is an integer $n = n(P) \in S$, such that $n\,\mathrm{coker}\,\mathbf{H}_P(\alpha) = 0$, for all epimorphisms α in $\mathcal{A}$;*

(vi) *The functor $\bar{\mathbf{H}}_P$ is exact.*

Proof. As we have seen before in 1.5, $\mathrm{Ext}^1_{\mathbb{Z}[S^{-1}]A}(P, A) \cong \mathbb{Z}[S^{-1}] \otimes \mathrm{Ext}^1_{\mathcal{A}}(A, B)$, so (i)⇔(ii) is immediate. Moreover, the implications (iv)⇒(iii), (iii)⇒(ii) and (iv)⇒(v) are obvious.

(ii)⇒(iv). We suppose that, for every $n \in S$, there exists $A_n \in \mathcal{A}$ such that $n\,\mathrm{Ext}_{\mathcal{A}}(P, A_n) \neq 0$. Then

$$\mathrm{Ext}^1_{\mathcal{A}}(P, \prod_{n\in S} A_n) \cong \prod_{n\in S} \mathrm{Ext}^1_{\mathcal{A}}(P, A_n)$$

is not S-torsion.

(v)⇒(vi) follows by 1.2.

(vi)⇒(iii) The exactness of $\bar{\mathbf{H}}_P$ implies that the group $\operatorname{coker}\mathbf{H}_P(\alpha)$ is bounded by an integer $n_\alpha > 0$ for any epimorphism α in $\mathcal{A}$. By the Ker-Coker Lemma, we deduce that $\mathrm{Ext}^1_{\mathcal{A}}(P, A)$ are bounded, for all $A \in \mathcal{A}$

We will say that the exact sequence in $\mathcal{A}$:

$$E : 0 \to K \stackrel{\alpha}{\to} L \stackrel{\beta}{\to} M \to 0$$

S-splits if it represents a splitting exact sequence in $\mathbb{Z}[S^{-1}]\mathcal{A}$ and, consequently, a S-monomorphism or a S-epimorphism *S-splits* if it splits in $\mathbb{Z}[S^{-1}]$Mod-R.

Lemma 2.3. *Let $E : 0 \to K \stackrel{\alpha}{\to} L \stackrel{\beta}{\to} M \to 0$ be an exact sequence in* Mod-R. *The following are equivalent:*

a) E S-splits;

b) There exists $\beta' : M \to L$ such that $\beta\beta' = n1_M$ for some integer $n \in S$;

c) There exists $\alpha' : L \to K$ such that $\alpha'\alpha = n1_L$ for some integer $n \in S$.

Proof. This statement is a consequence of the characterization of the split short exact sequences [5, Exercise 4.3.13], using the fact that, for some integer $n \in S$, the homomorphism n_A represents the identity of A in $\mathbb{Z}[S^{-1}]$Mod-R.

Corollary 2.4. *The following are equivalent for an R-module P:*

a) P is an almost projective;

b) Every epimorphism $M \to P \to 0$ in $\mathcal{A}$ splits in $\mathbb{Z}[S^{-1}]\mathcal{A}$;

*c) P is isomorphic in $\mathbb{Z}[S^{-1}]$*Mod-R *to a S-direct summand in a free module;*

d) There exist an integer $n \in S$, a family of R-homomorphisms $\varphi_i : P \to R$, $i \in I$, and $x_i \in P$, $i \in I$, such that $na = \sum_{i\in I} \varphi_i(a)x_i$ for all $a \in P$.

Proof. *a*) ⇒ *b*) If $\alpha : M \to P$ is an epimorphism and P is almost projective, then the kernel of $\bar{\mathbf{H}}_P(\alpha) : \mathrm{Hom}_{\mathcal{A}}(P, M) \to \mathrm{Hom}_{\mathcal{A}}(P, P)$ is bounded by an integer $n \in S$ and this shows that there exists $\alpha' : P \to M$ such that $\alpha\alpha' = n1_P$.

b) ⇒ *c*) is obvious.

c) ⇒ *d*) If P is S-isomorphic to a S-direct summand in $R^{(I)}$, let $\beta : R^{(I)} \to P$ be a S-epimorphism which splits in $\mathbb{Z}[S^{-1}]$Mod-R, and let $\varphi : P \to R^{(I)}$ be a homomorphism such that $\beta\phi = n1_P$. We choose a basis $(e_i)_{i\in I}$ in $R^{(I)}$. Denote by $\varphi_i : P \to R$ the composite homomorphisms $\pi_i\varphi$, where $\pi_i : R^{(I)} \to R$ are the canonical projections, and put $x_i = \beta(e_i)$, where $(e_i)_{i\in I}$ is the canonical basis in

$R^{(I)}$. Since, for every $a \in P$, $\varphi_i(a) = 0$ for almost all i, we may write $\varphi : P \to R$, $\varphi = \sum_{i\in I} \varphi_i$ and we obtain

$$na = \beta\varphi(a) = \sum_{i\in I}(\beta\varphi_i)(a) = \sum_{i\in I}(\varphi_i(a)\beta(e_i)) = \sum_{i\in I}(\varphi_i(a)x_i).$$

d) ⇒ *c*) The family $\varphi_i : P \to R$ induces a homomorphism $\varphi : P \to R^{(I)}$, while the correspondences $e_i \mapsto x_i$ give a homomorphism $\beta : R^{(I)} \to P$ which is an epimorphism in $\mathbb{Z}[S^{-1}]$Mod-R. Moreover, $\beta\varphi = n1_P$ showing that β splits in $\mathbb{Z}[S^{-1}]$Mod-R.

The implication *c*) ⇒ *a*) is a consequence of the following observation: the class of projective objects in an abelian category is closed with respect the direct summands and the canonical functor preserves the projective objects.

We shall say that the pair $((\varphi_i)_{i\in I}, (x_i)_{i\in I})$ is a *S-dual basis for P*. Observe that for a finite generated module we may suppose that I is a finite set.

Corollary 2.5. *If P is a S-almost projective right R-module, then $\mathbb{Z}[S^{-1}]\otimes P$ is a projective $\mathbb{Q}\otimes R$-module.*

Proof. We consider $((\varphi_i)_{i\in I}, (x_i)_{i\in I})$ a S-dual basis for P. Then there exists an integer $n \in S$ such that for every $m \in M$, we have $nm = \sum_{i\in I}\varphi_i(m)x_i$. Thus

$$(1\otimes\varphi_i, \frac{1}{n}\otimes x_i) \in \mathbb{Z}[S^{-1}]\otimes \mathrm{Hom}_{\mathbb{Z}[S^{-1}]\otimes R}(\mathbb{Z}[S^{-1}]\otimes M, \mathbb{Z}[S^{-1}]\otimes R)\times\mathbb{Z}[S^{-1}]\otimes M,$$

with $i \in I$, is a dual basis for the $\mathbb{Z}[S^{-1}]\otimes R$-module $\mathbb{Z}[S^{-1}]\otimes M$.

Remark 2.6. The converse of the previous statement is not true. Indeed, if A is a torsion abelian group which is not a bounded group, then it is not almost projective over $\mathbb{Z}$ (see corollary 2.8), but $0 = \mathbb{Z}[S^{-1}]\otimes A$ is a projective $\mathbb{Z}[S^{-1}]\otimes\mathbb{Z}$-module.

Proposition 2.7. *If R is a hereditary ring, then an R-module is S-almost projective if and only if it is a direct sum between a projective R-module and a R-module which is bounded as an abelian group.*

Proof. Let P be a S-almost projective R-module. If F is a free R-module and $\beta : F \to P$ is a R-epimorphism then β splits in $\mathbb{Z}[S^{-1}]$Mod-R, hence there exists $\beta' : P \to F$ such that $\beta\beta' = n1_P$ for some integer $n \in S$. Hence $\mathrm{Ker}(\beta')$ is bounded as an abelian group and it follows that P and $\mathrm{Im}(\beta')$ are isomorphic in $\mathbb{Z}[S^{-1}]$Mod-R. Observe that $\mathrm{Im}(\beta')$ is projective and it follows that the exact sequence $0 \to \mathrm{Ker}(\beta') \to P \to \mathrm{Im}(\beta') \to 0$ splits. The converse it obvious.

Corollary 2.8. *An abelian group is S-almost projective as $\mathbb{Z}$-module if and only if it is a direct sum between a free abelian group and a S-bounded group.*

In an analogous way we may prove

Proposition 2.9. *The following are equivalent for an object $I \in \mathcal{A}$:*

(i) *I is S-almost injective;*

(ii) *The group $\mathrm{Ext}_{\mathcal{A}}(A, I)$ is S-torsion for all $A \in \mathcal{A}$;*

(iii) *The group $\mathrm{Ext}^1_{\mathcal{A}}(A, I)$ is S-bounded, for all $A \in \mathcal{A}$;*

(iv) *There is an integer $n = n(I) \in S$, such that $n\,\mathrm{Ext}^1_{\mathcal{A}}(A, I) = 0$ for all $A \in \mathcal{A}$;*

(v) *There is an integer $n = n(I) \in S$, such that $n\,\mathrm{coker}\,\mathbf{H}^I(\alpha) = 0$, for all monomorphisms α in $\mathcal{A}$;*

(vi) *The functor $\bar{\mathbf{H}}^I$ is exact.*

Corollary 2.10. *An abelian group A is almost injective as an $\mathbb{Z}$-module if and only if $A \cong B \oplus D$ with B a bounded group and D a divisible group.*

For almost injective R-modules, as in the standard case we obtain an analogous statement to the Baer's criterion [8, Proposition I.6.5].

Proposition 2.11. *The R-module I is almost injective if and only if there exists an integer $n \in S$ such that for every right ideal U of R and for every R-homomorphism $\alpha : U \to R$, there exists $r \in R$ such that $n\alpha(x) = rx$ for all $x \in U$.*

3. **Almost flat modules**

In the end we give the interpretation for the almost-flat modules, introduced by Albrecht and Goeters in [1].

If A is a left R-module, and $\mathbf{T}_A = - \otimes_R A : \text{Mod-}R \to \mathcal{A}b$ is the tensor product functor, we shall denote by $\overline{\mathbf{T}}_A : \mathbb{Z}[S^{-1}]\text{Mod-}R \to \mathbb{Z}[S^{-1}]\mathcal{A}b$ the induced functor. We say that A is *S-almost flat* if and only if the functor $\overline{\mathbf{T}}_A$ is exact. We obtain the following characterization which shows that our definition is compatible with the definition of almost flat modules given in [1].

Proposition 3.1. *Let A be a left R-module. Then the following are equivalents:*

a) A is S-almost flat;

b) If $f : M \to N$ is a monomorphism in Mod-R*, then the kernel of the canonical homomorphism $f \otimes_R 1_A : M \otimes_R A \to N \otimes_R A$ is S-bounded;*

c) There exists $n \in S$ such that for every monomorphism $f : M \to N$ in Mod-R *we have $n\,\mathrm{Ker}(f \otimes_R 1_A) = 0$;*

d) If $M \in$ Mod-R, then $\mathrm{Tor}^1_R(M, A)$ is S-bounded;

e) There exists $n \in S$ such that $n\,\mathrm{Tor}^1_R(M, A) = 0$ for all $M \in$ Mod-R.

Proof. *a)* $\Rightarrow$ *b)* If the sequence $0 \to M \xrightarrow{f} N$ is exact in Mod-R, then $\mathbf{q}(f)$ is a monomorphism, and it follows that $\bar{\mathbf{T}}_A(\mathbf{q}(f)) = \mathbf{q}(\mathbf{T}_A(f))$ is a monomorphism in $\mathbb{Z}[S^{-1}]\mathcal{A}b$ showing that $\mathrm{Ker}(\mathbf{T}_A(f))$ is S-bounded.

b) $\Rightarrow$ *a)* Let $0 \to \mathbf{q}(L) \to \mathbf{q}(M) \to \mathbf{q}(N) \to 0$ be an exact sequence in $\mathbb{Z}[S^{-1}]$Mod-R. Then, from [4, Corollaire III.1], there exists an exact sequence $0 \to L' \to M' \to N' \to 0$ in Mod-R such that we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{q}(L) & \longrightarrow & \mathbf{q}(M) & \longrightarrow & \mathbf{q}(N) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{q}(L') & \longrightarrow & \mathbf{q}(M') & \longrightarrow & \mathbf{q}(N') & \longrightarrow & 0 \end{array}$$

in $\mathbb{Z}[S^{-1}]$Mod-R, the vertical arrows being isomorphisms. The short exact sequence $0 \to L' \to M' \to N' \to 0$ induces by hypothesis the exact sequence in $\mathcal{A}b$

$$0 \to B \to \mathbf{T}_A(L') \to \mathbf{T}_A(M') \to \mathbf{T}_A(N') \to 0,$$

with B a S-bounded group. This shows that in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bar{\mathbf{T}}_A(L) & \longrightarrow & \bar{\mathbf{T}}_A(M) & \longrightarrow & \bar{\mathbf{T}}_A(N) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bar{\mathbf{T}}_A(L') & \longrightarrow & \bar{\mathbf{T}}_A(M') & \longrightarrow & \bar{\mathbf{T}}_A(N') & \longrightarrow & 0 \end{array}$$

the rows are exact, and the vertical arrows are isomorphisms. It follows that the sequence $0 \to \bar{\mathbf{T}}_A(L) \to \bar{\mathbf{T}}_A(M) \to \bar{\mathbf{T}}_A(N) \to 0$ is exact, hence A is S-almost flat.

b) $\Rightarrow$ *d)* Let M be a right R-module, and let $0 \to K \xrightarrow{f} P \to M \to 0$ be an exact sequence with P projective. We apply $\mathbf{T}_A$ to obtain $Tor^1_R(M, A) \cong \mathrm{Ker}(\mathbf{T}_A(f))$. Observe that the last group is S-bounded, because f is a monomorphism.

d) $\Rightarrow$ *e)* The proof is similar with the proof of [1, Proposition 2.1]

$e) \Rightarrow c)$ follows from the fact that for every monomorphism $f : M \to N$, the group $\mathrm{Ker}(\mathbf{T}_A(f))$ is a homomorphic image of $\mathrm{Tor}^1_R(N/f(M), A)$. The implication $c) \Rightarrow b)$ is obvious.

Corollary 3.2. *An abelian group G is S-almost flat as an $\mathbb{Z}$-module if and only if $G = B \oplus H$ with B a S-bounded group and H a torsion free group*

Proof. If $G = B \oplus H$ with B a S-bounded group and H a torsion free group, then for every abelian group K we obtain $\mathrm{Tor}(K, G) \cong \mathrm{Tor}(K, B)$ and the last group is S-bounded.

Conversely, if $\mathrm{Tor}(K, G)$ is bounded by $n \in S$ for all $K \in \mathcal{A}b$ then then for every integer $m > 1$, using the isomorphism $\mathrm{Tor}(\mathbb{Z}(m), G) \cong G[m]$ ([3, Property 62(H)]), we obtain $nG[m] = 0$, proving that the torsion part of G is bounded by n. Therefore, G splits and its torsion part is a S-bounded group.

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