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UNIVALENCE CONDITIONS FOR CERTAIN INTEGRAL OPERATORS

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Abstract. In this paper the result of V. Pescar and S. Owa, on univalence conditions of integral operators, is extended to the case of n univalent functions. New results are presented in theorems 1 and 3.

1. **Introduction**

Let A be the class of functions f, which are analytic in the unit disc $U = \{z \in C; |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$ and let us denote with S the class of univalent functions.

Theorem A. [4] *If f is the univalent in U ,$\alpha \in C$ and $|\alpha| \leq \frac{1}{4}$ then the function*

$$G_\alpha(z) = \int_0^z [f'(t)]^\alpha \, dt$$

is univalent in U.

Theorem B. [3] *If the function $g \in S$,$\alpha \in C$, $|\alpha| \leq \frac{1}{4n}$ then the function defined by*

$$G_{\alpha,n}(z) = \int_0^z [f'(t^n)]^\alpha \, dt$$

is univalent in U for $n \in N^*$.

Theorem C. [2] *Let $\alpha, a \in C, \operatorname{Re}\alpha > 0$ and $f \in A$.*

If

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1$$

$(\forall)\, z \in U$ *then* $(\forall)\, \beta \in U, \operatorname{Re}\beta \geq \operatorname{Re}\alpha$, *the function*

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta-1} f'(t) \right]^{\frac{1}{\beta}}$$

is univalent.

Theorem D [1]. *If the function g is olomorphic in U and $|g(z)| < 1$ in U, then for all $\xi \in U$ and $z \in U$ the following inequalities hold*

$$\left|\frac{g(\xi) - g(z)}{1 - \overline{g(z)}g(\xi)}\right| \leq \left|\frac{\xi - z}{1 - \overline{z}\xi}\right| \qquad (*)$$

and

$$|g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2}$$

the equalities hold in case $g(z) = \varepsilon \frac{z+u}{1+\overline{u}z}$ where $|\varepsilon| = 1$ and $|u| < 1$.

Remark E [1]. For $z = 0$, from inequality (*) we obtain for every $\xi \in U$

$$\left|\frac{g(\xi) - g(0)}{1 - \overline{g(0)}g(\xi)}\right| \leq |\xi|$$

and, hence

$$|g(\xi)| \leq \frac{|\xi| + |g(0)|}{1 + |g(0)|\,|\xi|}$$

Considering $g(0) = a$ and $\xi = z$, then

$$|g(z)| \leq \frac{|z| + |a|}{1 + |a|\,|z|}$$

for all $z \in U$.

Lemma F (Schwartz). *If the function g is olomorphic in $U, g(0) = 0$ and $|g(z)| \leq 1\,(\forall)\, z \in U$ then result:*

$|g(z)| \leq |z|, (\forall)\, z \in U$ *and* $|g'(0)| \leq 1$ *the equalities hold in case* $g(z) \leq \varepsilon z$ *where* $|\varepsilon| = 1$.

Theorem G [5]. *Let $\alpha, \gamma \in C, \operatorname{Re}\alpha = a > 0, g \in A, g(z) = z + b_2 z^2 + \dots$.*

If

$$\left|\frac{g''(z)}{g'(z)}\right| \leq \frac{1}{n}, (\forall)\, z \in U$$

and

$$|\gamma| \leq \frac{n + 2a}{2} \cdot \left(\frac{n + 2a}{n}\right)^{\frac{n}{2a}}$$

then $(\forall), \beta \in C, \operatorname{Re}\beta \geq a$, the function

$$G_{\beta,\gamma,n}(z) = \left\{\beta \int_0^z t^{\beta-1} \cdot [g'(t^n)]^{\gamma}\, dt\right\}^{\frac{1}{\beta}}$$

is univalent $(\forall)\, n \in N^* \setminus \{1\}$.

Theorem H [5]. *Let* $\alpha, \gamma \in C, \operatorname{Re}\alpha = b > 0, g \in A, g(z) = z + a_2 z^2 + \ldots$.

If

$$\left| \frac{g''(z)}{g'(z)} \right| \leq 1, (\forall)\, z \in U$$

and

$$|\gamma| \leq \frac{1}{\max\limits_{|z| \leq 1} \left[\frac{1-|z|^{2c}}{c} \cdot |z| \cdot \frac{|z|+2|a_2|}{1+2|a_2||z|} \right]}$$

then $(\forall)\, \beta \in C, \operatorname{Re}\beta \geq b,$ *the function*

$$G_{\beta,\gamma}(z) = \left\{ \beta \int\limits_0^z t^{\beta-1} \cdot [g'(t)]^{\gamma}\, dt \right\}^{\frac{1}{\beta}}$$

is univalent.

2. Main results

Theorem 1. *Let* $\alpha, \gamma_i \in C, (\forall)\, i = \overline{1,p}, \operatorname{Re}\alpha = a \geq 0, f_i \in A, f_i(z) = z + a_2^i z^2 + \ldots, (\forall)\, i = \overline{1,p}$.

If

$$\left| \frac{f_i''(z)}{f_i'(z)} \right| \leq \frac{1}{n}, (\forall)\, z \in U, i = \overline{1,p} \tag{1}$$

$$\frac{|\gamma_1| + \ldots + |\gamma_p|}{|\gamma_1 \cdot \ldots \cdot \gamma_p|} < 1 \tag{2}$$

and

$$|\gamma_1 \cdot \ldots \cdot \gamma_p| \leq \frac{n+2a}{2} \cdot \left(\frac{n+2a}{n} \right)^{\frac{n}{2a}} \tag{3}$$

then $(\forall)\, \beta \in C, \operatorname{Re}\beta \geq a$,*the function*

$$G(z) = \left\{ \beta \int\limits_0^z t^{\beta-1} \cdot [f_1'(t^n)]^{\gamma_1} \cdot \ldots \cdot [f_p'(t^n)]^{\gamma_p}\, dt \right\}^{\frac{1}{\beta}}$$

is univalent $(\forall)\, n \in N^* \setminus \{1\}$.

Proof. Let

$$h(z) \int\limits_0^z [f_1'(t^n)]^{\gamma_1} \cdot \ldots \cdot [f_p'(t^n)]^{\gamma_p}\, dt$$

$$p(z) = \frac{1}{|\gamma_1 \cdot \ldots \cdot \gamma_p|} \cdot \frac{h''(z)}{h'(z)}$$

where $|\gamma_1 \cdot \ldots \cdot \gamma_p|$ satisfies (3).

We have

$$p(z) = \frac{\gamma_1}{\left|\gamma_1 \cdot ... \cdot \gamma_p\right|} \cdot \frac{n \cdot z^{n-1} \cdot f_1''(z^n)}{f_1'(z^n)} + ... + \frac{\gamma_p}{\left|\gamma_1 \cdot ... \cdot \gamma_p\right|} \cdot \frac{n \cdot z^{n-1} \cdot f_p''(z^n)}{f_p'(z^n)}$$

Applying the relations (1) and (2) we obtain:

$$|p(z)| \leq \frac{|\gamma_1|}{\left|\gamma_1 \cdot ... \cdot \gamma_p\right|} \cdot \left|\frac{n \cdot z^{n-1} \cdot f_1''(z^n)}{f_1'(z^n)}\right| + ... + \frac{|\gamma_p|}{\left|\gamma_1 \cdot ... \cdot \gamma_p\right|} \cdot \left|\frac{n \cdot z^{n-1} \cdot f_p''(z^n)}{f_p'(z^n)}\right| \leq$$

$$\leq \frac{|\gamma_1| + ... + |\gamma_p|}{\left|\gamma_1 \cdot ... \cdot \gamma_p\right|} < 1$$

Considering Schwartz's lemma we have:

$$\frac{1}{\left|\gamma_1 \cdot ... \cdot \gamma_p\right|} \cdot \left|\frac{h''(z)}{h'(z)}\right| \leq \left|z^{n-1}\right| \leq |z| \Leftrightarrow \left|\frac{h''(z)}{h'(z)}\right| \leq \left|\gamma_1 \cdot ... \cdot \gamma_p\right| \cdot \left|z^{n-1}\right| \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{1-|z|^{2a}}{a}\right) \cdot \left|\frac{zh''(z)}{h'(z)}\right| \leq \left|\gamma_1 \cdot ... \cdot \gamma_p\right| \cdot \left(\frac{1-|z|^{2a}}{a}\right) \cdot |z^n| \tag{4}$$

Let's the function $Q : [0,1] \to R, Q(x) = \left(\frac{1-x^{2a}}{a}\right) \cdot x^n, x = |z|.$

We have

$$Q(x) \leq \frac{n+2a}{2} \cdot \left(\frac{n+2a}{n}\right)^{\frac{n}{2a}} (\forall)\, x \in [0,1]$$

According to the conditions (3) and (4) we obtain:

$$\left(\frac{1-|z|^{2a}}{a}\right) \cdot \left|\frac{zh''(z)}{h'(z)}\right| \leq 1$$

so, according to Theorem C, G is univalent.

Corollary 2. *Let* $\alpha, \beta, \gamma, \delta \in C, \operatorname{Re}\delta = a > 0, f, g \in A, f(z) = z + a_2 z^2 + ..., g(z) = z + b_2 z^2 + ...$.

If

$$\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{1}{n}, (\forall)\, z \in U$$

$$\left|\frac{g''(z)}{g'(z)}\right| \leq \frac{1}{n}, (\forall)\, z \in U$$

$$\frac{1}{|\alpha|} + \frac{1}{|\beta|} < 1$$

and

$$|\alpha\beta| \leq \frac{n+2a}{2} \cdot \left(\frac{n+2a}{n}\right)^{\frac{n}{2a}}$$

then $(\forall)\,\gamma \in C, \operatorname{Re}\gamma \geq a$,*the function*

$$D_{\alpha,\beta,\gamma,n}(z) = \left\{ \gamma \int_0^z t^{\gamma-1} \cdot [f'(t^n)]^{\alpha} \cdot [g'(t^n)]^{\beta} \, dt \right\}^{\frac{1}{\gamma}}$$

is univalent $(\forall)\, n \in N^* \setminus \{1\}$.

Proof. In Theorem 1, we consider $p = 2, f_1 = f, f_2 = g, \gamma_1 = \alpha, \gamma_2 = \beta, \gamma = \beta$.

Remark. If in Theorem 1, we consider $p =, f_1 = g, \gamma_1 = \gamma, \gamma = \beta$, we obtained Theorem G.

Theorem 3. *Let* $\alpha, \gamma_i \in C, (\forall)\, i = \overline{1,n}, \operatorname{Re}\alpha = b > 0, f_i \in A, f_i(z) = z + a_2^i z^2 + ..., (\forall)\, i = \overline{1,n}$.

If

$$\left| \frac{f_i''(z)}{f_i'(z)} \right| \leq 1, (\forall)\, z \in U, i = \overline{1,n} \tag{5}$$

$$\frac{|\gamma_1| + ... + |\gamma_n|}{|\gamma_1 \cdot ... \cdot \gamma_n|} < 1 \tag{6}$$

and

$$|\gamma_1 \cdot ... \cdot \gamma_n| \leq \frac{1}{\max\limits_{|z| \leq 1} \left[\frac{1-|z|^{2b}}{b} \cdot |z| \cdot \frac{|z|+2|c|}{1+2|c||z|} \right]} \tag{7}$$

then $(\forall)\, \beta \in C, \operatorname{Re}\beta \geq b$,*the function*

$$H(z) = \left\{ \beta \int_0^z t^{\beta-1} \cdot [f_1'(t)]^{\gamma_1} \cdot ... \cdot [f_n'(t)]^{\gamma_n} \, dt \right\}^{\frac{1}{\beta}}$$

is univalent $(\forall)\, n \in N$.

Proof. Let

$$h(z) = \int_0^z [f_1'(t)]^{\gamma_1} \cdot ... \cdot [f_n'(t)]^{\gamma_n} \, dt$$

$$p(z) = \frac{1}{|\gamma_1 \cdot ... \cdot \gamma_n|} \cdot \frac{h''(z)}{h'(z)}$$

where $|\gamma_1 \cdot ... \cdot \gamma_n|$ satisfies (7).

We have

$$p(z) = \frac{\gamma_1}{|\gamma_1 \cdot ... \cdot \gamma_n|} \cdot \frac{f_1''(z)}{f_1'(z)} + ... + \frac{\gamma_n}{|\gamma_1 \cdot ... \cdot \gamma_n|} \cdot \frac{f_n''(z)}{f_n'(z)}$$

p is olomorph and $|\gamma_1 \cdot ... \cdot \gamma_n|$ satisfies the relation (7) implies $|p(z)| < 1$ according to (5) and (6).

$$p(0) = \frac{\gamma_1 a_2^1 + ... + \gamma_n a_2^n}{|\gamma_1 \cdot ... \cdot \gamma_n|} = c$$

$$|p(z)| \leq \frac{|z| + 2|c|}{1 + 2|c||z|}, (\forall) z \in U \Leftrightarrow \frac{1}{|\gamma_1 \cdot ... \cdot \gamma_n|} \cdot \left| \frac{zh''(z)}{h'(z)} \right| \leq \frac{|z| + 2|c|}{1 + 2|c||z|}, (\forall) z \in U \Leftrightarrow$$

$$\left(\frac{1 - |z|^{2b}}{b} \right) \cdot \left| \frac{zh''(z)}{h'(z)} \right| \leq |\gamma_1 \cdot ... \cdot \gamma_n| \cdot \max_{|z| \leq 1} \left[\frac{1 - |z|^{2b}}{b} \cdot |z| \cdot \frac{|z| + 2|c|}{1 + 2|c||z|} \right] \leq 1, (\forall) z \in U$$

so, according to Theorem C, H is univalent $(\forall), n \in N$.

Corollary 4. *Let* $\alpha, \beta, \gamma, \delta \in C, \operatorname{Re} \delta = c > 0, f, g \in A, f(z) = z + a_2 z^2 + ..., g(z) = z + b_2 z^2 + ...$.

If

$$\left| \frac{f''(z)}{f'(z)} \right| \leq 1, (\forall) z \in U$$

$$\left| \frac{g''(z)}{g'(z)} \right| \leq 1, (\forall) z \in U$$

$$\frac{1}{|\alpha|} + \frac{1}{|\beta|} < 1$$

and

$$|\alpha\beta| \leq \frac{1}{\max\limits_{|z| \leq 1} \left[\frac{1 - |z|^{2c}}{c} \cdot |z| \cdot \frac{|z| + 2\frac{|\alpha a_2 + \beta b_2|}{|\alpha\beta|}}{1 + 2\frac{|\alpha a_2 + \beta b_2|}{|\alpha\beta|}|z|} \right]}$$

then $(\forall) \gamma \in C, \operatorname{Re} \gamma \geq c$, *the function*

$$F_{\alpha,\beta,\gamma}(z) = \left\{ \gamma \int_0^z t^{\gamma - 1} \cdot [f'(t)]^{\alpha} \cdot [g'(t)]^{\beta} dt \right\}^{\frac{1}{\gamma}}$$

is univalent.

Proof. In Theorem 3, we consider $p = 2, f_1 = f, f_2 = g, \gamma_1 = \alpha, \gamma_2 = \beta$.

Remark. If in Theorem 3, we consider $p = 1, f_1 = g, \gamma_1 = \gamma, \gamma = \beta$, we obtained Theorem H.

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