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SIMPLE SUBALGEBRAS OF GROUP GRADED ALGEBRAS

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Abstract. We study the situation when the 1-component A_1 of a G-graded $\mathcal{O}$-algebra A has an $\mathcal{O}$-simple subalgebra $S \simeq M_n(\mathcal{O})$. We prove that the centralizer $C_A(S)$ of S is a graded subalgebra of A, and that there is a graded Morita equivalence between A and $C_A(S)$. This generalizes a theorem of L. Puig.

1. **Introduction**

Let G be a finite group and let $\mathcal{O}$ be a commutative local noetherian ring, complete with respect to the $J(\mathcal{O})$-adic topology, and such that the residue field $k = \mathcal{O}/J(\mathcal{O})$ is algebraically closed of characteristic $p > 0$. All $\mathcal{O}$-algebras are assumed to be finitely generated and free as $\mathcal{O}$-modules.

If $A = \bigoplus_{g \in G} A_g$ and $B = \bigoplus_{g \in G} B_g$ are two G-graded $\mathcal{O}$-algebras, then recall that the $\mathcal{O}$-algebra homomorphism $f : A \to B$ is called *G-graded* if $f(A_g) \subseteq B_g$ for all $g \in G$. A subalgebra C of A is a *graded subalgebra* if for any $c = \sum_{g \in G} c_g \in C$, the homogeneous component c_g also belongs to C for all $g \in G$. In this case we have that $C = \bigoplus_{g \in G} C_g$, where $C_g = C \cap A_g$.

An $\mathcal{O}$-algebra S is called *$\mathcal{O}$-simple* if is isomorphic to $\mathrm{End}_O(V)$ for some free $\mathcal{O}$-module V, that is, if S isomorphic to a matrix algebra $M_n(\mathcal{O})$ over $\mathcal{O}$ (where n is the dimension of V).

The centralizer of the subalgebra S in A is, by definition, the subalgebra

$$C_A(S) = \{a \in A \mid as = sa \text{ for all } s \in S\}.$$

If B is a G-graded $\mathcal{O}$-algebra, then the matrix algebra $A = M_n(B)$ is a G-graded algebra, where for each $g \in G$, A_g consists of matrices with entries in B_g. The A_1 has a subalgebra S isomorphic to $M_n(\mathcal{O})$, and there is an isomorphism $C_A(S) \simeq B$ of G-graded algebras, mapping an element $a \in C_A(S)$ to $eae = ea = ae$, where e is the matrix having 1 in the top left corner and 0 elsewhere. Moreover, there is an isomorphism $A \simeq S \otimes_{\mathcal{O}} C_A(S)$ of G-graded algebras, and there is a graded Morita equivalence between A and B (see Section 3 below).

In this note we consider the converse situation. We assume that $A = \bigoplus_{g\in G} A_g$ is a G-graded algebra and $S \simeq M_n(\mathcal{O})$ is an $\mathcal{O}$-simple subalgebra of A_1, and we show that there is a graded Morita equivalence between A and $C_A(S)$. This generalizes a theorem of L. Puig [2] (see also [3, Sections 1.7 and 1.9]. For notions and results on graded algebras and graded Morita equivalences we refer to [1].

2. **Simple subalgebras**

In this section $A = \bigoplus_{g\in G} A_g$ is a G-graded $\mathcal{O}$-algebra and $S \simeq \mathrm{End}_{\mathcal{O}}(L)$ be a G-graded $\mathcal{O}$-simple subalgebra of A_1 with $1_S = 1_A$. Let $C_A(S)$ be the centralizer of S and let e be a primitive idempotent of S. The next results are generalizations of [3, Propositions 7.5 and 7.6].

Proposition 2.1 *With the above notations and assumptions, the following statements hold.*

a) *$C_A(S)$ is a G-graded subalgebra of S.*

b) *There is an isomorphism of G-graded $\mathcal{O}$-algebras given by*

$$\phi : S \otimes_{\mathcal{O}} C_A(S) \to A, \qquad \phi(s \otimes a) = sa.$$

c) *There is an isomorphism of G-graded $\mathcal{O}$-algebras given by*

$$\eta : C_A(S) \to eAe, \qquad \eta(a) = ea = ae = eae.$$

Proof. a) We know that $C_A(S)$ is a subalgebra of A. We have to prove that $C_A(S)$ is G-graded subalgebra. For any $a = \sum_{g\in G} a_g \in A$, if $a \in C_A(S)$, then we have $as = sa$ for all $s \in S$. It follows that $\sum_{g\in G} a_g s = \sum_{g\in G} s a_g$. Since $S \subseteq A_1$, $a_g s = s a_g$ for all $s \in S$ and $g \in G$. This means that $a_g \in C_A(S)$ for all $g \in G$.

b) We know from the proof of [3, Proposition 7.5] that ϕ is an isomorphism of $\mathcal{O}$-algebras and that the map

$$\psi : A \to S \otimes_{\mathcal{O}} C_A(S), \quad \psi(a) = \sum_{u,v\in U} (u^{-1}ev \otimes \sum_{w\in U} (euav^{-1}e)^w)$$

is $\mathcal{O}$-algebra homomorphism, which is the inverse of ϕ. Here U denotes a finite set of invertible elements of S satisfying $1_S = \sum_{u\in U} e^u$ (recall that all the primitive idempotents of S are conjugate). We only have to verify that ϕ and ψ are grade-preserving.

Because A is a G-graded algebra, we have that $S \otimes_{\mathcal{O}} C_A(S)$ is also G-graded, with components $(S \otimes_{\mathcal{O}} C_A(S))_g = S \otimes_{\mathcal{O}} C_A(S)_g$. If $s \otimes a_g \in S \otimes_{\mathcal{O}} C_A(S)_g$, we have that $\phi(s \otimes a_g) = sa_g$ belongs to $SA_g \subseteq A_1A_g = A_g$, hence $\phi(S \otimes_{\mathcal{O}} C_A(S)_g) \subseteq A_g$. Finally, if $a_g \in A_g$ then

$$\psi(a_g) = \sum_{u,v\in U} (u^{-1}ev \otimes \sum_{w\in U} (eua_g v^{-1}e)^w) \in S \otimes_{\mathcal{O}} C_A(S)_g$$

since $U \subset A_1$, so $\psi(A_g) \subseteq S \otimes_{\mathcal{O}} C_A(S)_g$.

c) We know that $C_A(S)$ and eAe are isomorphic as $\mathcal{O}$-algebras. We have to prove they are isomorphic as G-graded algebras. For all $a_g \in C_A(S)_g$ we have $\eta(a_g) = ea_ge$ belongs to eA_ge, so $\eta(C_A(S)_g) \subseteq eA_ge$. Consequently η is G-graded. Similarly, the inverse of η, given by $eae \mapsto \sum_{u\in U}(eae)^u$ is a G-graded map, so the proposition is proved.

Proposition 2.2. Let M be a G-graded A-module. Then there is an isomorphism of G-graded A-modules given by

$$\phi : Se \otimes_{\mathcal{O}} eM \to M, \qquad \phi(s \otimes m) = sm.$$

Proof. Since M is a G-graded A-module and $e \in S \subseteq A_1$, we have that eM is a G-graded eAe-submodule of M, hence eM is a G-graded $C_A(S)$-module via the isomorphism η of Proposition 2.1 c). Consequently $Se \otimes_{\mathcal{O}} eM$ is a G-graded $S \otimes_{\mathcal{O}} C_A(S)$-module. We know that ϕ is homomorphism of A-modules. Letting $1_A = 1_S = \sum_{u\in U} e^u$ be a primitive decomposition of the identity in S, consider the map

$$\psi : M \to Se \otimes_{\mathcal{O}} eM, \qquad \psi(m) = \sum_{u\in U} u^{-1}e \otimes eum,$$

where U is a finite set of invertible elements of S.

We are going to show that ψ is the inverse of ϕ and that both maps are grade-preserving. First we have that

$$\begin{aligned}(\phi\circ\psi)(m) &= \phi(\sum_{u\in U}u^{-1}e\otimes eum)=\sum_{u\in U}\phi(u^{-1}e\otimes eum)\\ &= \sum_{u\in U}u^{-1}eum=\sum_{u\in U}e^{u}m=m,\end{aligned}$$

because $1_S=1_A=\sum_{u\in U}e^u$.

On the other hand let $m\in M$ and let $s^{-1}et$ be a basis element of S, where $s,t\in U$. Then we have

$$\begin{aligned}(\psi\circ\phi)(s^{-1}ete\otimes em) &= \psi(s^{-1}etem)=\sum_{u\in U}u^{-1}e\otimes eus^{-1}etem\\ &= \sum_{u\in U}u^{-1}e\otimes u(u^{-1}eu)(s^{-1}es)s^{-1}tem\\ &= s^{-1}e\otimes etem=s^{-1}ete\otimes em,\end{aligned}$$

where we have used that $e^ue^s=0$ unless $u=s$.

For all $s\otimes m_g\in Se\otimes_{\mathcal{O}}eM_g$, we have that $\phi(s\otimes m_g)=sm_g$ belongs to $SM_g\subseteq M_g$, so $\phi(Se\otimes_{\mathcal{O}}eM_g)\subseteq M_g$. Similarly, if $m_g\in M_g$, the $\psi(m_g)$ belongs to $Se\otimes_{\mathcal{O}}eM_g$ since $U\subset A_1$ and $e\in A_1$.

3. **A Morita equivalence**

We keep the notations and assumptions of the preceding section. The following result is a generalization to the case of G-graded algebras of [2, Theorem 3].

Theorem 3.1. *The algebras A and $C_A(S)$ are graded Morita equivalent.*

Proof. Since A is isomorphic to $S\otimes_{\mathcal{O}}C_A(S)$ as G-graded algebras, it is enough to prove the following statement. Let C be an $\mathcal{O}$-algebra and let $S\simeq \mathrm{End}_{\mathcal{O}}(L)$ be an $\mathcal{O}$-simple algebra. Then $S\otimes_{\mathcal{O}}C$ is graded Morita equivalent to C. Indeed, consider the functor

$$F: C-\mathrm{mod}\to S\otimes_{\mathcal{O}}C-\mathrm{mod},\quad F(M)=L\otimes_{\mathcal{O}}M.$$

Observe that if $M=\bigoplus_{x\in G}M_x$ is a G-graded C-module, then $F(M)$ is a G-graded $S\otimes_{\mathcal{O}}C$-module with components $F(M)_x=L\otimes_{\mathcal{O}}M_x$ for all $x\in G$. Moreover, if $M(g)$ is the g-th suspension of M (where $M(g)_x=M_{xg}$ for all $x\in G$), then

$F(M(g)) = F(M)(g)$. Therefore, the restriction of F gives a graded functor F^{gr} : $C-\mathrm{gr} \to S \otimes_{\mathcal{O}} C-\mathrm{gr}$, which clearly commutes with the grade forgetting functor. It remains to prove that F is an equivalence of categories. Observe that $L \simeq Se$, where e is a primitive idempotent of S. By replacing A with $A \otimes_{\mathcal{O}} C$ and e with $e \otimes 1$, Proposition 2.2 shows that any $S \otimes_{\mathcal{O}} C$-module is naturally isomorphic to a module of the form $L \otimes_{\mathcal{O}} M$, where M is a C-module. This immediately implies that F is an equivalence.

Remark 3.2. Alternatively, we could have used the isomorphism $C_A(S) \simeq eAe$ of G-graded algebras. Since $1_A = 1_S = \sum_{u \in U} e^u$, we have that $AeA = A$. Consequently, the G-graded bimodules Ae and e induce a graded Morita equivalence between A and eAe.

References

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