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PICARD PAIRS AND WEAKLY PICARD PAIRS OF OPERATORS

ALINA SÎNTĂMĂRIAN

Abstract. The purpose of this paper is to introduce the notions of Picard pair, c-Picard pair, weakly Picard pair and c-weakly Picard pair of operators and to present examples for these notions. We also study the data dependence of the common fixed points set of c-weakly Picard pairs of operators.

1. **Introduction**

Let (X, d) be a metric space. Further on we shall need the following notations

$$P(X) := \{\ Y \mid \emptyset \neq Y \subseteq X\ \}$$

$$P_{cl}(X) := \{\ Y \mid Y \in P(X) \text{ and } Y \text{ is a closed set }\}$$

and the following functionals

$$D : P(X) \times P(X) \to \mathbb{R}_+,\ D(A, B) = \inf\ \{\ d(a, b) \mid a \in A,\ b \in B\ \},$$

$$H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ H(A, B) = \max\ \left\{ \sup_{a \in A} D(a, B),\ \sup_{b \in B} D(b, A) \right\}.$$

Let $f_1, f_2 : X \to X$ be two operators. We denote by G_{f_1} the graph of f_1, by F_{f_1} the fixed points set of f_1 and by $(CF)_{f_1,f_2}$ the common fixed points set of f_1 and f_2.

The purpose of this paper is to study the following problems:

Problem 1.1. *Let (X, d) be a metric space and $f_1, f_2 : X \to X$ be two operators. Determine the metric conditions which imply that (f_1, f_2) is a (weakly) Picard pair of operators or (and) f_1, f_2 are (weakly) Picard operators.*

Problem 1.2. *Let (X, d) be a metric space and $f_1, f_2, g_1, g_2 : X \to X$ be four operators such that $(CF)_{f_1,f_2}, (CF)_{g_1,g_2} \neq \emptyset$. We suppose that there exists $\eta > 0$ with*

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the property that for each $x \in X$*, there are* $i_x, j_x \in \{1,2\}$ *so that* $d(f_{i_x}(x), g_{j_x}(x)) \leq \eta$*. In these conditions estimate the Pompeiu-Hausdorff distance* $H((CF)_{f_1,f_2}, (CF)_{g_1,g_2})$*.*

Throughout the paper we follow the terminology and the notations from Rus [7], [8] and Rus-Mureşan [9], [10].

2. **Picard pairs and weakly Picard pairs of operators**

Definition 2.1. [Rus [6], [7], [8]] *Let* (X,d) *be a metric space. An operator* $f : X \to X$ *is a* Picard operator (*briefly* P. o.) *iff there exists* $x^* \in X$ *such that* $F_f = \{x^*\}$ *and* $(f^n(x_0))_{n\in\mathbb{N}}$ *converges to* x^**, for all* $x_0 \in X$*.*

Let (X,d) be a metric space. We say that a P. o. $f : X \to X$ is a *c-Picard operator* ($c \in [0,+\infty[$) (briefly *c-P. o.*) iff the following condition is satisfied

$$d(x, x^*) \leq c\, d(x, f(x)),$$

for each $x \in X$, where x^* is the unique fixed point of f.

Definition 2.2. [Rus [6], [7], [8]] *Let* (X,d) *be a metric space. An operator* $f : X \to X$ *is a* weakly Picard operator (*briefly* w. P. o.) *iff for each* $x_0 \in X$*, the sequence* $(f^n(x_0))_{n\in\mathbb{N}}$ *converges and its limit is a fixed point of* f*.*

For examples of P. o. and w. P. o. see for instance Rus [6], [7], [8].

Let (X,d) be a metric space and $f : X \to X$ be a w. P. o.. We consider the operator $f^\infty : X \to F_f$, defined as follows

$$f^\infty(x) = \lim_{n\to\infty} f^n(x),$$

for each $x \in X$.

Definition 2.3. [Rus-Mureşan [10]] *Let* (X,d) *be a metric space and* $f : X \to X$ *be a w. P. o.. We say that* f *is a* c-weakly Picard operator ($c \in [0,+\infty[$) (*briefly* c-w. P. o.) *iff the following condition is satisfied*

$$d(x, f^\infty(x)) \leq c\, d(x, f(x)),$$

for each $x \in X$*.*

Examples of c-w. P. o. are given in Rus-Mureşan [10].

Definition 2.4. *Let* (X,d) *be a metric space and* $f_1, f_2 : X \to X$ *be two operators. We say that the pair of operators* (f_1, f_2) *is a* Picard pair of operators (*briefly* P. p. o.) *iff there exists* $x^* \in X$ *such that* $(CF)_{f_1,f_2} = \{x^*\}$ *and for each* $x \in X$ *and for* *every* $y \in \{f_1(x), f_2(x)\}$*, the sequence* $(x_n)_{n\in\mathbb{N}}$ *defined as follows:* $x_0 = x$*,* $x_1 = y$ *and* $x_{2n-1} = f_i(x_{2n-2})$*,* $x_{2n} = f_j(x_{2n-1})$*, for each* $n \in \mathbb{N}^*$*, where* $i, j \in \{1, 2\}$*, with* $i \neq j$*, converges to* x^**.*

The sequence $(x_n)_{n\in\mathbb{N}}$ *is, by definition, a* sequence of successive approximations for the pair (f_1, f_2), starting from (x_0, x_1).

Definition 2.5. *Let* (X, d) *be a metric space and* $f_1, f_2 : X \to X$ *be two operators which form a P. p. o.. We say that* (f_1, f_2) *is a* c-Picard pair of operators ($c \in [0, +\infty[$) (*briefly* c-P. p. o.) *iff the following condition is satisfied*

$$d(x, x^*) \leq c\, d(x, y),$$

for each $(x, y) \in G_{f_1} \cup G_{f_2}$*, where* x^* *is the unique common fixed point of* f_1 *and* f_2*.*

Definition 2.6. *Let* (X, d) *be a metric space and* $f_1, f_2 : X \to X$ *be two operators. We say that the pair of operators* (f_1, f_2) *is a* weakly Picard pair of operators (*briefly* w. P. p. o.) *iff for each* $x \in X$ *and for every* $y \in \{f_1(x), f_2(x)\}$*, there exists a sequence* $(x_n)_{n\in\mathbb{N}}$ *such that:*

(i) $x_0 = x$, $x_1 = y$;

(ii) $x_{2n-1} = f_i(x_{2n-2})$ *and* $x_{2n} = f_j(x_{2n-1})$*, for each* $n \in \mathbb{N}^*$*, where* $i, j \in \{1, 2\}$*, with* $i \neq j$;

(iii) *the sequence* $(x_n)_{n\in\mathbb{N}}$ *is convergent and its limit is a common fixed point of* f_1 *and* f_2*.*

Definition 2.7. *Let* (X, d) *be a metric space and* $f_1, f_2 : X \to X$ *be two operators which form a w. P. p. o.. Then we consider the multivalued operator* $(f_1, f_2)^\infty : G_{f_1} \cup G_{f_2} \to P((CF)_{f_1,f_2})$ *as follows: for each* $(x, y) \in G_{f_1} \cup G_{f_2}$*, we define* $(f_1, f_2)^\infty(x, y) = \{\ z \in (CF)_{f_1,f_2} \mid$ *there exists a sequence of successive approximations for the pair* (f_1, f_2)*, starting from* (x, y)*, that converges to* $z\ \}$*.*

Definition 2.8. *Let* (X, d) *be a metric space and* $f_1, f_2 : X \to X$ *be two operators which form a w. P. p. o.. We say that* (f_1, f_2) *is a* c-weakly Picard pair of operators ($c \in [0, +\infty[$) (*briefly* c-w. P. p. o.) *iff there exists a selection* $f_{1,2}^\infty$ *of* $(f_1, f_2)^\infty$ *such that*

$$d(x, f_{1,2}^\infty(x, y)) \leq c\, d(x, y),$$

for each $(x, y) \in G_{f_1} \cup G_{f_2}$*.*

Remark 2.1. *It is obvious that a P. p. o. is a w. P. p. o. and a c-P. p. o. is a c-w. P. p. o..*

Further on we shall give some examples of c-P. p. o. and c-w. P. p. o..

Theorem 2.1. *Let (X, d) be a complete metric space and $f_1, f_2 : X \to X$ be two operators for which there exists $a \in [0, 1/2[$ such that*

$$d(f_1(x), f_2(y)) \leq a\ [d(x, f_1(x)) + d(y, f_2(y))],$$

for each $x, y \in X$.

Then $F_{f_1} = F_{f_2} = \{x^*\}$, (f_1, f_2) is c-P. p. o. and f_1 and f_2 are c-P. o., with $c = (1 - a)/(1 - 2a)$.

Proof. The conclusion follows immediately from Kannan's theorem [3] and from the Theorem 2 given by Rus in [5]. □

Theorem 2.2. *Let (X, d) be a complete metric space and $f_1, f_2 : X \to X$ be two operators for which there exist $a, b \in \mathbb{R}_+$, with $a + b < 1$ such that*

$$d(f_1(x), f_2(y)) \leq a\ d(x, f_1(x)) + b\ d(y, f_2(y)),$$

for each $x, y \in X$.

Then $F_{f_1} = F_{f_2} = \{x^*\}$ and (f_1, f_2) is c-P. p. o., with $c = (1 - \min\{a, b\})/[1 - (a + b)]$.

Theorem 2.3. *Let (X, d) be a complete metric space and $f_1, f_2 : X \to X$ be two operators. We suppose that there exist $\alpha, \beta, \gamma \in \mathbb{R}_+$, with $\alpha + 2\beta + 2\gamma < 1$ such that*

$$d(f_1(x), f_2(y)) \leq \alpha\ d(x, y) + \beta\ [d(x, f_1(x)) + d(y, f_2(y))] + \gamma\ [d(x, f_2(y)) + d(y, f_1(x))],$$

for each $x, y \in X$.

Then $F_{f_1} = F_{f_2} = \{x^*\}$ and (f_1, f_2) is c-P. p. o., with $c = [1 - (\beta + \gamma)]/[1 - (\alpha + 2\beta + 2\gamma)]$.

Proof. The fact that $F_{f_1} = F_{f_2} = \{x^*\}$ follows from a theorem given by Rus in [4].

In order to prove the second part of the conclusion we shall take again the proof.

Let $i, j \in \{1, 2\}$, with $i \neq j$. Let $x_0 \in X$ and we take $x_{2n-1} = f_i(x_{2n-2})$, $x_{2n} = f_j(x_{2n-1})$, for each $n \in \mathbb{N}^*$.

We have

$$d(x_1, x_2) = d(f_i(x_0), f_j(x_1)) \le$$

$$\le \alpha d(x_0, x_1) + \beta[d(x_0, f_i(x_0)) + d(x_1, f_j(x_1))] + \gamma[d(x_0, f_j(x_1)) + d(x_1, f_i(x_0))] =$$

$$= \alpha d(x_0, x_1) + \beta[d(x_0, x_1) + d(x_1, x_2)] + \gamma d(x_0, x_2) \le$$

$$\le \alpha d(x_0, x_1) + \beta[d(x_0, x_1) + d(x_1, x_2)] + \gamma[d(x_0, x_1) + d(x_1, x_2)]$$

and hence

$$d(x_1, x_2) \le (\alpha + \beta + \gamma)/[1 - (\beta + \gamma)]\ d(x_0, x_1).$$

Similarly, we have that

$$d(x_2, x_3) \le (\alpha + \beta + \gamma)/[1 - (\beta + \gamma)]\ d(x_1, x_2).$$

By induction we get that

$$d(x_n, x_{n+1}) \le \left[\frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)}\right]^n d(x_0, x_1),$$

for each $n \in \mathbb{N}$.

This implies that $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence, because (X, d) is a complete metric space. The limit of the sequence $(x_n)_{n\in\mathbb{N}}$ is the unique common fixed point x^* of f_1 and f_2.

We have

$$d(x_n, x^*) \le \left[\frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)}\right]^n \frac{1 - (\beta + \gamma)}{1 - (\alpha + 2\beta + 2\gamma)}\ d(x_0, x_1),$$

for each $n \in \mathbb{N}$.

For $n = 0$, we obtain

$$d(x_0, x^*) \le [1 - (\beta + \gamma)]/[1 - (\alpha + 2\beta + 2\gamma)]\ d(x_0, f_i(x_0)).$$

So, we can assert that (f_1, f_2) is a c-P. p. o., with $c = [1-(\beta+\gamma)]/[1-(\alpha+2\beta+2\gamma)]$.
□

Remark 2.2. *If we take* $\alpha = \beta = 0$ *in the metric condition of the Theorem* 2.3, *then the part which affirms that* $F_{f_1} = F_{f_2} = \{x^*\}$ *is a result given by Chatterjea in* [1] *and we have that* (f_1, f_2) *is c-P. p. o., with* $c = (1 - \gamma)/(1 - 2\gamma)$.

Theorem 2.4. *Let* (X, d) *be a complete metric space and* $f_1, f_2 : X \to X$ *be two operators for which there exist* $a_1, \ldots, a_5 \in \mathbb{R}_+$, *with* $a_1 + a_2 + a_3 + 2 \max \{a_4, a_5\} < 1$ *such that*

$$d(f_1(x), f_2(y)) \leq a_1\ d(x, y) + a_2\ d(x, f_1(x)) + a_3\ d(y, f_2(y)) +$$

$$+ a_4\ d(x, f_2(y)) + a_5\ d(y, f_1(x)),$$

for each $x, y \in X$.

Then $F_{f_1} = F_{f_2} = \{x^*\}$ *and* (f_1, f_2) *is c-P. p. o., with* $c = (1 - l)^{-1}$, *where* $l = \max\ \{(a_1 + a_2 + a_4)/[1 - (a_3 + a_4)], (a_1 + a_3 + a_5)/[1 - (a_2 + a_5)]\}$.

Proof. The proof is made similarly with that of the Theorem 2.3. □

Theorem 2.5. *Let* (X, d) *be a complete metric space and* $f_1, f_2 : X \to X$ *be two operators. We suppose that there exists* $a \in [0, 1[$ *such that*

$$d(f_1(x), f_2(y)) \leq a\ \max\ \{d(x, y), d(x, f_1(x)), d(y, f_2(y)),$$

$$1/2\ [d(x, f_2(y)) + d(y, f_1(x))]\},$$

for each $x, y \in X$.

Then $F_{f_1} = F_{f_2} = \{x^*\}$ *and* (f_1, f_2) *is c-P. p. o., with* $c = (1 - a)^{-1}$.

Proof. The fact that $F_{f_1} = F_{f_2} = \{x^*\}$ follows from a theorem given by Ćirić in [2]. For the second part of the conclusion, the proof is made similarly with that of the Theorem 2.3. □

Theorem 2.6. *Let* (X, d) *be a complete metric space and* $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ *be a continuous function which satisfies the following two conditions*:

(i_φ) φ *is monoton increasing in each variable*;

(ii_φ) $\varphi(t, t, t, 2t, 0) \leq t$, $\varphi(t, t, t, 0, 2t) \leq t$ *and* $\varphi(t, 0, 0, t, t) \leq t$, *for each* $t > 0$.

Let $f_1, f_2 : X \to X$ *be two operators for which there exists* $a \in [0, 1[$ *such that*

$$d(f_1(x), f_2(y)) \leq a\ \varphi(d(x, y), d(x, f_1(x)), d(y, f_2(y)), d(x, f_2(y)), d(y, f_1(x))),$$

for each $x, y \in X$.

Then $F_{f_1} = F_{f_2} = \{x^*\}$ *and* (f_1, f_2) *is c-P. p. o., with* $c = (1 - a)^{-1}$.

Proof. The proof is made similarly with that of the Theorem 2.3, taking into account the properties of the function φ. □

Remark 2.3. *It is an open question if the operators* f_1 *and* f_2 *from the Remark* 2.2*, the Theorems* 2.2*,* 2.3*,* 2.4*,* 2.5 *or* 2.6 *are P. o..*

Theorem 2.7. *Let* (X, d) *be a complete metric space and* $f_1, f_2 : X \to X$ *be two continuous operators. We suppose that there exist* $a_1, a_2 \in [0, 1[$ *such that for each* $i, j \in \{1, 2\}$*, with* $i \neq j$ *we have*

$$d(f_i(x), f_j(f_i(x))) \leq a_i \ d(x, f_i(x)),$$

for each $x \in X$*.*

Then $F_{f_1} = F_{f_2} \in P_{cl}(X)$ *and* (f_1, f_2) *is c-w. P. p. o., with* $c = (1 - \max\{a_1, a_2\})^{-1}$*.*

Proof. We show in the beginning that $F_{f_1} = F_{f_2}$. Let $x^* \in F_{f_1}$. Then we have

$$d(x^*, f_2(x^*)) = d(f_1(x^*), f_2(f_1(x^*))) \leq a_1 \ d(x^*, f_1(x^*)) = 0.$$

So $x^* \in F_{f_2}$ and thus we are able to write that $F_{f_1} \subseteq F_{f_2}$. Analogously we get that $F_{f_2} \subseteq F_{f_1}$. Hence $F_{f_1} = F_{f_2}$.

It is not difficult to see that F_{f_1} and F_{f_2} are closed sets. In order to prove that let $i \in \{1, 2\}$ and $x_n \in F_{f_i}$, for each $n \in \mathbb{N}$, with the property that $x_n \to x^*$, as $n \to \infty$. From $x_n = f_i(x_n)$, for each $n \in \mathbb{N}$ and taking into account the fact that f_i is continuous we get, by letting n to tend to infinity, that $x^* = f_i(x^*)$, i. e. $x^* \in F_{f_i}$. So F_{f_i} is a closed set.

Further on we shall prove that $(CF)_{f_1,f_2} \neq \emptyset$. Let $i, j \in \{1, 2\}$, with $i \neq j$. Let $x_0 \in X$ and we put $x_{2n-1} = f_i(x_{2n-2})$, $x_{2n} = f_j(x_{2n-1})$, for each $n \in \mathbb{N}^*$. We have

$$d(x_1, x_2) = d(f_i(x_0), f_j(x_1)) = d(f_i(x_0), f_j(f_i(x_0))) \leq$$

$$\leq a_i \ d(x_0, f_i(x_0)) = a_i \ d(x_0, x_1).$$

Similarly, we have that

$$d(x_2, x_3) \leq a_j \ d(x_1, x_2).$$

We put $a = \max\{a_1, a_2\}$. By induction we get that

$$d(x_n, x_{n+1}) \leq a^n \ d(x_0, x_1),$$

for each $n \in \mathbb{N}$.

This implies that $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence, because (X,d) is a complete metric space. Let $x^* = \lim_{n\to\infty} x_n$. From $x_{2n-1} = f_i(x_{2n-2})$, $x_{2n} = f_j(x_{2n-1})$, for each $n \in \mathbb{N}^*$ and taking into account the fact that f_1 and f_2 are continuous, it follows that $x^* \in (CF)_{f_1,f_2}$. So $(CF)_{f_1,f_2} = F_{f_1} = F_{f_2} \neq \emptyset$. By an easy calculation we have

$$d(x_n, x^*) \leq a^n/(1-a)\ d(x_0, x_1),$$

for each $n \in \mathbb{N}$.

For $n = 0$ we get

$$d(x_0, x^*) \leq (1-a)^{-1}\ d(x_0, x_1).$$

Therefore (f_1, f_2) is a c-w. P. p. o., where $c = (1 - \max\{a_1, a_2\})^{-1}$. □

Remark 2.4. *It is an open question if the operators f_1 and f_2 from the Theorem* 2.7 *are w. P. o..*

3. Data dependence of the common fixed points set of c-weakly Picard pairs of operators

Theorem 3.1. *Let (X,d) be a metric space and $f_1, f_2, g_1, g_2 : X \to X$ be four operators. We suppose that:*

*(*i*) (f_1, f_2) is a c_f-w. P. p. o. and (g_1, g_2) is a c_g-w. P. p. o.;*

*(*ii*) there exists $\eta > 0$ such that for each $x \in X$, there are $i_x, j_x \in \{1,2\}$ so that*

$$d(f_{i_x}(x), g_{j_x}(x)) \leq \eta.$$

Then

$$H\left((CF)_{f_1,f_2}, (CF)_{g_1,g_2}\right) \leq \eta\ \max\{c_f, c_g\}.$$

Proof. It is not difficult to see that

$$H\left((CF)_{f_1,f_2}, (CF)_{g_1,g_2}\right) \leq \max\left\{ \sup_{x\in(CF)_{g_1,g_2}} d(x, f_{1,2}^{\infty}(x, f_{i_x}(x))), \sup_{x\in(CF)_{f_1,f_2}} d(x, g_{1,2}^{\infty}(x, g_{j_x}(x))) \right\}.$$

If $x \in (CF)_{g_1,g_2}$, then we have

$$d(x, f_{1,2}^{\infty}(x, f_{i_x}(x))) \leq c_f \; d(x, f_{i_x}(x)) = c_f \; d(g_{j_x}(x), f_{i_x}(x)) \leq c_f \; \eta.$$

If $x \in (CF)_{f_1,f_2}$, then we get

$$d(x, g_{1,2}^{\infty}(x, g_{j_x}(x))) \leq c_g \; d(x, g_{j_x}(x)) = c_g \; d(f_{i_x}(x), g_{j_x}(x)) \leq c_g \; \eta.$$

From these, using the following lemma (see [8])

Lemma 3.1. *Let (X, d) be a metric space and $A, B \in P(X)$. We suppose that there exists $\eta \in \mathbb{R}$, $\eta > 0$ such that:*

(i) for each $a \in A$, there exists $b \in B$ so that $d(a, b) \leq \eta$,

(ii) for each $b \in B$, there exists $a \in A$ so that $d(b, a) \leq \eta$.

Then $H(A, B) \leq \eta$.

We obtain the conclusion of the theorem. □

Further on we shall give some consequences of the abstract result given in Theorem 3.1.

Theorem 3.2. *Let (X, d) be a complete metric space and $f_1, f_2, g_1, g_2 : X \to X$ be four operators. We suppose that:*

(i_f) there exists $a_f \in [0, 1/2[$ such that

$$d(f_1(x), f_2(y)) \leq a_f \; [d(x, f_1(x)) + d(y, f_2(y))],$$

for each $x, y \in X$;

(i_g) there exists $a_g \in [0, 1/2[$ such that

$$d(g_1(x), g_2(y)) \leq a_g \; [d(x, g_1(x)) + d(y, g_2(y))],$$

for each $x, y \in X$;

(ii) there exists $\eta > 0$ such that for each $x \in X$, there are $i_x, j_x \in \{1, 2\}$ so that

$$d(f_{i_x}(x), g_{j_x}(x)) \leq \eta.$$

Then $F_{f_1} = F_{f_2} = \{x_f^*\}$, $F_{g_1} = F_{g_2} = \{x_g^*\}$ and

$$d(x_f^*, x_g^*) \leq \eta \; (1 - a)/(1 - 2a),$$

where $a = \max \{a_f, a_g\}$.

Proof. From the Theorem 2.1 we have that $F_{f_1} = F_{f_2} = \{x_f^*\}$ and that (f_1, f_2) is c_f-P. p. o., with $c_f = (1 - a_f)/(1 - 2a_f)$. From the same theorem we also have that $F_{g_1} = F_{g_2} = \{x_g^*\}$ and that (g_1, g_2) is c_g-P. p. o., with $c_g = (1 - a_g)/(1 - 2a_g)$. The fact that $d(x_f^*, x_g^*) \leq \eta\ (1-a)/(1-2a)$ follows immediately from the Remark 2.1 and the Theorem 3.1. □

Theorem 3.3. *Let (X, d) be a complete metric space and $f_1, f_2, g_1, g_2 : X \to X$ be four operators. We suppose that:*

(i_f) there exist $a_f, b_f \in \mathbb{R}_+$, with $a_f + b_f < 1$ such that

$$d(f_1(x), f_2(y)) \leq a_f\ d(x, f_1(x)) + b_f\ d(y, f_2(y)),$$

for each $x, y \in X$;

(i_g) there exist $a_g, b_g \in \mathbb{R}_+$, with $a_g + b_g < 1$ such that

$$d(g_1(x), g_2(y)) \leq a_g\ d(x, g_1(x)) + b_g\ d(y, g_2(y)),$$

for each $x, y \in X$;

(ii) there exists $\eta > 0$ such that for each $x \in X$, there are $i_x, j_x \in \{1, 2\}$ so that

$$d(f_{i_x}(x), g_{j_x}(x)) \leq \eta.$$

Then $F_{f_1} = F_{f_2} = \{x_f^*\}$, $F_{g_1} = F_{g_2} = \{x_g^*\}$ and

$$d(x_f^*, x_g^*) \leq \eta\ \max\ \{c_f, c_g\},$$

where $c_f = (1-\min\ \{a_f, b_f\})/[1-(a_f+b_f)]$ and $c_g = (1-\min\ \{a_g, b_g\})/[1-(a_g+b_g)]$.

Proof. From the Theorem 2.2 we have that $F_{f_1} = F_{f_2} = \{x_f^*\}$ and that (f_1, f_2) is c_f-P. p. o.. From the same theorem we also have that $F_{g_1} = F_{g_2} = \{x_g^*\}$ and that (g_1, g_2) is c_g-P. p. o.. Now, the fact that $d(x_f^*, x_g^*) \leq \eta\ \max\ \{c_f, c_g\}$ follows immediately from the Remark 2.1 and the Theorem 3.1. □

Theorem 3.4. *Let (X, d) be a complete metric space and $f_1, f_2, g_1, g_2 : X \to X$ be four operators. We suppose that:*

(i_f) there exist $\alpha_f, \beta_f, \gamma_f \in \mathbb{R}_+$, with $\alpha_f + 2\beta_f + 2\gamma_f < 1$ such that

$$d(f_1(x), f_2(y)) \leq \alpha_f\ d(x, y) + \beta_f\ [d(x, f_1(x)) + d(y, f_2(y))]+$$

$$+\gamma_f\ [d(x, f_2(y)) + d(y, f_1(x))],$$

for each $x, y \in X$;

(i_g) *there exist* $\alpha_g, \beta_g, \gamma_g \in \mathbb{R}_+$, *with* $\alpha_g + 2\beta_g + 2\gamma_g < 1$ *such that*

$$d(g_1(x), g_2(y)) \le \alpha_g\ d(x,y) + \beta_g\ [d(x, g_1(x)) + d(y, g_2(y))]+$$

$$+\gamma_g\ [d(x, g_2(y)) + d(y, g_1(x))],$$

for each $x, y \in X$;

(ii) *there exists* $\eta > 0$ *such that for each* $x \in X$, *there are* $i_x, j_x \in \{1,2\}$ *so that*

$$d(f_{i_x}(x), g_{j_x}(x)) \le \eta.$$

Then $F_{f_1} = F_{f_2} = \{x_f^*\}$, $F_{g_1} = F_{g_2} = \{x_g^*\}$ *and*

$$d(x_f^*, x_g^*) \le \eta\ \max\ \{c_f, c_g\},$$

where $c_f = [1-(\beta_f+\gamma_f)]/[1-(\alpha_f+2\beta_f+2\gamma_f)]$ *and* $c_g = [1-(\beta_g+\gamma_g)]/[1-(\alpha_g+2\beta_g+2\gamma_g)]$.

Proof. From the Theorem 2.3 we have that $F_{f_1} = F_{f_2} = \{x_f^*\}$ and that (f_1, f_2) is c_f-P. p. o.. From the same theorem we also have that $F_{g_1} = F_{g_2} = \{x_g^*\}$ and that (g_1, g_2) is c_g-P. p. o.. The fact that $d(x_f^*, x_g^*) \le \eta\ \max\ \{c_f, c_g\}$ follows immediately from the Remark 2.1 and the Theorem 3.1. □

Theorem 3.5. *Let* (X, d) *be a complete metric space and* $f_1, f_2, g_1, g_2 : X \to X$ *be four operators. We suppose that*:

(i_f) *there exist* $a_1^f, \ldots, a_5^f \in \mathbb{R}_+$, *with* $a_1^f + a_2^f + a_3^f + 2\max\ \{a_4^f, a_5^f\} < 1$ *such that*

$$d(f_1(x), f_2(y)) \le a_1^f\ d(x,y) + a_2^f\ d(x, f_1(x)) + a_3^f\ d(y, f_2(y))+$$

$$+a_4^f\ d(x, f_2(y)) + a_5^f\ d(y, f_1(x)),$$

for each $x, y \in X$;

(i_g) *there exist* $a_1^g, \ldots, a_5^g \in \mathbb{R}_+$, *with* $a_1^g + a_2^g + a_3^g + 2\max\ \{a_4^g, a_5^g\} < 1$ *such that*

$$d(g_1(x), g_2(y)) \le a_1^g\ d(x,y) + a_2^g\ d(x, g_1(x)) + a_3^g\ d(y, g_2(y))+$$

$$+a_4^g\ d(x, g_2(y)) + a_5^g\ d(y, g_1(x)),$$

for each $x, y \in X$;

(ii) *there exists* $\eta > 0$ *such that for each* $x \in X$*, there are* $i_x, j_x \in \{1, 2\}$ *so that*

$$d(f_{i_x}(x), g_{j_x}(x)) \leq \eta.$$

Then $F_{f_1} = F_{f_2} = \{x_f^*\}$, $F_{g_1} = F_{g_2} = \{x_g^*\}$ *and*

$$d(x_f^*, x_g^*) \leq \eta \ \max\{c_f, c_g\},$$

where $c_f = (1 - l_f)^{-1}$*, with* $l_f = \max\{(a_1^f + a_2^f + a_4^f)/[1 - (a_3^f + a_4^f)], (a_1^f + a_3^f + a_5^f)/[1 - (a_2^f + a_5^f)]\}$ *and* $c_g = (1 - l_g)^{-1}$*, with* $l_g = \max\{(a_1^g + a_2^g + a_4^g)/[1 - (a_3^g + a_4^g)], (a_1^g + a_3^g + a_5^g)/[1 - (a_2^g + a_5^g)]\}$.

Proof. From the Theorem 2.4 we have that $F_{f_1} = F_{f_2} = \{x_f^*\}$ and that (f_1, f_2) is c_f-P. p. o.. From the same theorem we also have that $F_{g_1} = F_{g_2} = \{x_g^*\}$ and that (g_1, g_2) is c_g-P. p. o.. Now, the fact that $d(x_f^*, x_g^*) \leq \eta \ \max\{c_f, c_g\}$ follows from the Remark 2.1 and the Theorem 3.1. □

Theorem 3.6. *Let* (X, d) *be a complete metric space and* $f_1, f_2, g_1, g_2 : X \to X$ *be four operators. We suppose that:*

(i_f) *there exists* $a_f \in [0, 1[$ *such that*

$$d(f_1(x), f_2(y)) \leq a_f \ \max\{d(x, y), d(x, f_1(x)), d(y, f_2(y)),$$
$$1/2\ [d(x, f_2(y)) + d(y, f_1(x))]\},$$

for each $x, y \in X$;

(i_g) *there exists* $a_g \in [0, 1[$ *such that*

$$d(g_1(x), g_2(y)) \leq a_g \ \max\{d(x, y), d(x, g_1(x)), d(y, g_2(y)),$$
$$1/2\ [d(x, g_2(y)) + d(y, g_1(x))]\},$$

for each $x, y \in X$;

(ii) *there exists* $\eta > 0$ *such that for each* $x \in X$*, there are* $i_x, j_x \in \{1, 2\}$ *so that*

$$d(f_{i_x}(x), g_{j_x}(x)) \leq \eta.$$

Then $F_{f_1} = F_{f_2} = \{x_f^*\}$, $F_{g_1} = F_{g_2} = \{x_g^*\}$ *and*

$$d(x_f^*, x_g^*) \leq \eta \ (1 - \max\{a_f, a_g\})^{-1}.$$

Proof. From the Theorem 2.5 we have that $F_{f_1} = F_{f_2} = \{x_f^*\}$ and that (f_1, f_2) is c_f-P. p. o., with $c_f = (1 - a_f)^{-1}$. From the same theorem we also have that $F_{g_1} = F_{g_2} = \{x_g^*\}$ and that (g_1, g_2) is c_g-P. p. o., with $c_g = (1 - a_g)^{-1}$. The fact that $d(x_f^*, x_g^*) \leq \eta\ (1 - \max\ \{a_f, a_g\})^{-1}$ follows immediately from the Remark 2.1 and the Theorem 3.1. □

Theorem 3.7. *Let (X, d) be a complete metric space and $\varphi_f, \varphi_g : \mathbb{R}_+^5 \to \mathbb{R}_+$ be two continuous functions which satisfy the following conditions*:

($\mathrm{i}_{\varphi_{f,g}}$) *$\varphi_f$ and φ_g are monoton increasing in each variable*;

($\mathrm{ii}_{\varphi_{f,g}}$) *$\varphi_f(t,t,t,2t,0) \leq t$, $\varphi_f(t,t,t,0,2t) \leq t$ and $\varphi_f(t,0,0,t,t) \leq t$, for each $t > 0$ and $\varphi_g(t,t,t,2t,0) \leq t$, $\varphi_g(t,t,t,0,2t) \leq t$ and $\varphi_g(t,0,0,t,t) \leq t$, for each $t > 0$.*

Let $f_1, f_2, g_1, g_2 : X \to X$ be four operators. We suppose that:

(i_f) *there exists $a_f \in [0, 1[$ such that*

$$d(f_1(x), f_2(y)) \leq a_f\ \varphi_f(d(x,y), d(x, f_1(x)), d(y, f_2(y)), d(x, f_2(y)), d(y, f_1(x))),$$

for each $x, y \in X$;

(i_g) *there exists $a_g \in [0, 1[$ such that*

$$d(g_1(x), g_2(y)) \leq a_g\ \varphi_g(d(x,y), d(x, g_1(x)), d(y, g_2(y)), d(x, g_2(y)), d(y, g_1(x))),$$

for each $x, y \in X$;

(ii) *there exists $\eta > 0$ such that for each $x \in X$, there are $i_x, j_x \in \{1, 2\}$ so that*

$$d(f_{i_x}(x), g_{j_x}(x)) \leq \eta.$$

Then $F_{f_1} = F_{f_2} = \{x_f^*\}$, $F_{g_1} = F_{g_2} = \{x_g^*\}$ and

$$d(x_f^*, x_g^*) \leq \eta\ (1 - \max\ \{a_f, a_g\})^{-1}.$$

Proof. From the Theorem 2.6 we have that $F_{f_1} = F_{f_2} = \{x_f^*\}$ and that (f_1, f_2) is c_f-P. p. o., with $c_f = (1 - a_f)^{-1}$. From the same theorem we also have that $F_{g_1} = F_{g_2} = \{x_g^*\}$ and that (g_1, g_2) is c_g-P. p. o., with $c_g = (1 - a_g)^{-1}$. Now, the fact that $d(x_f^*, x_g^*) \leq \eta\ (1 - \max\ \{a_f, a_g\})^{-1}$ follows immediately from the Remark 2.1 and the Theorem 3.1. □

Theorem 3.8. *Let (X, d) be a complete metric space and $f_1, f_2, g_1, g_2 : X \to X$ be four operators. We suppose that:*

(i_f) there exist $a_1^f, a_2^f \in [0, 1[$ such that for each $k, l \in \{1, 2\}$, with $k \neq l$ we have

$$d(f_k(x), f_l(f_k(x))) \leq a_k^f \ d(x, f_k(x)),$$

for each $x \in X$;

(i_g) there exist $a_1^g, a_2^g \in [0, 1[$ such that for each $k, l \in \{1, 2\}$, with $k \neq l$ we have

$$d(g_k(x), g_l(g_k(x))) \leq a_k^g \ d(x, g_k(x)),$$

for each $x \in X$;

(ii) there exists $\eta > 0$ such that for each $x \in X$, there are $i_x, j_x \in \{1, 2\}$ so that

$$d(f_{i_x}(x), g_{j_x}(x)) \leq \eta.$$

Then

$$H((CF)_{f_1,f_2}, (CF)_{g_1,g_2}) \leq \eta \ (1 - \max \ \{a_1^f, a_2^f, a_1^g, a_2^g\})^{-1}.$$

Proof. From the Theorem 2.7 we have that (f_1, f_2) is c_f-w. P. p. o., with $c_f = (1-\max \{a_1^f, a_2^f\})^{-1}$ and that (g_1, g_2) is c_g-w. P. p. o., with $c_g = (1-\max \{a_1^g, a_2^g\})^{-1}$. The conclusion follows from the Theorem 3.1. □

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Faculty of Mathematics and Computer Science,
Babeş-Bolyai University, 3400 Cluj-Napoca, Romania

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