STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 1, March 2002

THE φ-CATEGORY OF SOME PAIRS OF PRODUCTS OF MANIFOLDS

CORNEL PINTEA

Abstract. In this paper we will show that in certain topological conditions on the manifold M, the φ-category of the pairs

$(P_n(\mathbf{R}) \times M, T^a \times \mathbf{R}^{m-a}), (P_n(\mathbf{R}) \times M, T^a \times S^{m-a})$

is infinite for suitable choices of the numbers m, n, a.

1. Introduction

Let us first recall that the φ-category of a pair (M, N) of smooth manifolds is defined as

$$\varphi(M, N) = \min\{\#C(f) \mid f \in C^\infty(M, N)\},$$

where $C(f)$ denotes the critical set of the smooth mapping $f : M \to N$ and $\#C(f)$ its cardinality. For more details, see for instance [AnPi].

In the previous papers [Pi1], [Pi3] is studied the φ-category of the pairs $(P_n(\mathbf{R}), \mathbf{R}^m), (P_n(\mathbf{R}), S^m), (P_n(\mathbf{R}), T^a \times \mathbf{R}^{m-a})$ and is proved that it is infinite for suitable choices of the numbers m, n, a.

Using those results as well as some others, in this paper we will show the same think for some pairs of the form $(P_n(\mathbf{R}) \times M, T^a \times \mathbf{R}^{m-a}), (P_n(\mathbf{R}) \times M, T^a \times S^{m-a})$.

2. Some useful results

In this section we will recall some results proved in some various previous papers and which we are going to use in the next sections.

Theorem 2.1. ([Pi1]) *Let M, N be compact connected differentiable manifolds having the same dimension m. In these conditions the following statements are true:*

(i) If $m \geq 3$ and $\pi_1(N)$ has no subgroup isomorphic with $\pi_1(M)$, then $\varphi(M, N) \geq \aleph_0$;

(ii) If $m \geq 4$ and $\pi_q(M) \not\cong \pi_q(N)$ for some $q \in \{2, 3, \ldots, m-2\}$, then $\varphi(M, N) \geq \aleph_0$.

If G, H are two groups, then the *algebraic* φ*- category*of the pair (G, H) is defined as

$$\varphi_{alg}(G, H) = \min\{[H : \operatorname{Im} f] \mid f \in \operatorname{Hom}(G, H)\}.$$

Recall that for an abelian group G the subset $t(G)$ of all elements of finite order forms a subgroup of G called the *torsion subgroup*.

Proposition 2.2. ([Pi2]) *If G,H are finitely generated abelian groups such that* rank*[G/t(G)]*<rank*[H/t(H)], then* $\varphi_{alg}(G, H) \geq \aleph_0$

Theorem 2.3. ([Pi2]) *Let* M^m, N^n *be compact connected differential manifolds such that* $m \geq n \geq 2$*. If* $\varphi_{alg}(\pi(M), \pi(N)) \geq \aleph_0$*, then* $\varphi(M, N) \geq \aleph_0$*.*

Theorem 2.4. ([Pi3]) *If* M *is a smooth manifold and* n *is a natural number such that* $\dim M < n$*, then* $\varphi(M, \mathbf{R}^n) = \varphi(M, S^n)$*.*

Theorem 2.5. ([Pi3]) *If n is a natural number such that n+1 and n+2 are not powers of 2, then we have*

$$\varphi(P_n(\mathbf{R}), S^m) = \varphi(P_n(\mathbf{R}), \mathbf{R}^m) \geq \aleph_0 \quad \text{if } n < m \leq 2^{[log_2 n]+1} - 2$$
$$\varphi(P_n(\mathbf{R}), S^m) = \varphi(P_n(\mathbf{R}), \mathbf{R}^m) = 0 \quad \text{if } m \geq 2n - 1.$$

Theorem 2.6. ([Pi3]) *If* M^n, N^n, P *are differentiable manifolds such that* $\pi(P)$ *is a torsion group and* $\pi(N)$ *is a free torsion group and* $p : M \to N$ *is a differentiable covering mapping, then* $\varphi(P, M) = \varphi(P, N)$*.*

Corollary 2.7. ([Pi3]) *If M is a differentiable mapping such that* $\pi(M)$ *is a torsion group, then* $\varphi(M, \mathbf{R}^n) = \varphi(M, T^a \times \mathbf{R}^{n-a})$*, for any* $a \in \{1, \ldots, n-1\}$*. In particular, for* $a = n$ *we get that* $\varphi(M, \mathbf{R}^n) = \varphi(M, T^n)$*.*

Theorem 2.8. ([Pi4]) *If n is a natural number such that n+1 and n+2 are not powers of 2, then we have*

$$\varphi(P_n(\mathbf{R}), \mathbf{R}^m) \geq \aleph_0 \quad \text{if } n < m \leq 2^{[log_2 n]+1} - 2$$
$$\varphi(P_n(\mathbf{R}), \mathbf{R}^m) = 0 \quad \text{if } m \geq 2n - 1.$$

3. Main results

In this section we will see the announced topological conditions on the manifold M in order that the φ-category of the pairs $(P_n(\mathbf{R}) \times M, T^a \times \mathbf{R}^{m-a}), (P_n(\mathbf{R}) \times M, T^a \times S^{m-a})$ to be infinite.

Theorem 3.1. *If M,N,P are differentiable manifolds such that* $\dim M \leq \dim N \leq \dim P$ *and M is injectively immersable in N, then* $\varphi(M, P) \leq \varphi(N, P)$.

Proof. Let $j : M \to N$ be an injective immersion and $f : N \to P$ be a differential mapping. Recall that if $\alpha : X \to Y$ is a morphism of vector spaces (linear mapping) then $\dim X = \dim Ker\alpha + \dim Im\alpha$. Further on we have successively:

$$x \in C(f \circ j) \Leftrightarrow rank_x(f \circ j) < \dim M \Leftrightarrow \dim Imd(f \circ j)_x < \dim M \Leftrightarrow$$

$$\Leftrightarrow \dim M - \dim Kerd(f \circ j)_x < \dim M \Leftrightarrow \dim Ker[(df)_{j(x)} \circ (dj)_x] > 0 \Rightarrow$$

$$\Rightarrow \dim Ker(df)_{j(x)} > 0 \Leftrightarrow \dim N - \dim Im(df)_{j(x)} > 0 \Leftrightarrow$$

$$\Leftrightarrow \dim Im(df)_{j(x)} < \dim N \Leftrightarrow rank_{j(x)} f < \dim N \Leftrightarrow j(x) \in C(f).$$

Therefore we showed that $j[C(f \circ j)] \subseteq C(f)$, which implies that

$$\#C(f \circ j) = \#j[C(f \circ j)] \leq \#C(f),$$

that is $\varphi(M, P) \leq \#C(f \circ j) \leq \#C(f)$. The last inequalities holds for any differential mapping $f : N \to P$, which means that

$$\varphi(M, P) \leq \varphi(N, P).\Box$$

Theorem 3.2. *If n is a natural number such that n+1, n+2 are not powers of 2 and M is a differential manifold such that* $\pi(M)$ *is a torsion group, then we have*

(i) $\varphi(P_n(\mathbf{R}) \times M, T^a \times \mathbf{R}^{m-a}) \geq \aleph_0$ *if* $n + \dim M \leq m \leq 2^{[log_2 n]+1} - 2$, $\forall\, a \in \{1, \ldots, m-1\}$;

(ii) $\varphi(P_n(\mathbf{R}) \times M, T^a \times \mathbf{R}^{m-a}) = 0$ *if* $m \geq 2(n + \dim M)$ *and M is a compact manifold.*

Proof. (i) First of all observe that, according to corollary 2.7, $\varphi(P_n(\mathbf{R}) \times M, T^a \times \mathbf{R}^{m-a}) = \varphi(P_n(\mathbf{R}) \times M, \mathbf{R}^m)$. Because $P_n(\mathbf{R})$ can be embedded in $P_n(\mathbf{R}) \times M$ it follows, according to theorem 3.1, that $\varphi(P_n(\mathbf{R}) \times M, \mathbf{R}^m) \geq \varphi(P_n(\mathbf{R}), \mathbf{R}^m)$. But in the given hypothesis we get that $\varphi(P_n(\mathbf{R}), \mathbf{R}^m) \geq \aleph_0$, because of theorem 2.8, that is we have

$$\varphi(P_n(\mathbf{R}) \times M, T^a \times \mathbf{R}^{m-a}) = \varphi(P_n(\mathbf{R}) \times M, \mathbf{R}^m) \geq \varphi(P_n(\mathbf{R}), \mathbf{R}^m) \geq \aleph_0.$$

(ii) Follows easily from the equality $\varphi(P_n(\mathbf{R}) \times M, T^a \times \mathbf{R}^{m-a}) = \varphi(P_n(\mathbf{R}) \times M, \mathbf{R}^m)$ and from the Whitney's embedding theorem.$\Box$

Theorem 3.3. *If n is a natural number such that n+1, n+2 are not powers of 2 and M is a differential manifold, then we have*

(*i*) $\varphi(P_n(\mathbf{R}) \times M, S^m) = \varphi(P_n(\mathbf{R}) \times M, \mathbf{R}^m) \geq \aleph_0$
if $n + \dim M < m \leq 2^{[log_2 n]+1} - 2$;

(*ii*) $\varphi(P_n(\mathbf{R}) \times M, S^m) = \varphi(P_n(\mathbf{R}) \times M, \mathbf{R}^m) = 0$ *if* $m \geq 2(n + \dim M)$

and M is a compact manifold.

Proof. (*i*) First of all observe that, according to theorem 2.1, $\varphi(P_n(\mathbf{R}) \times M, S^m) = \varphi(P_n(\mathbf{R}) \times M, \mathbf{R}^m)$. Because $P_n(\mathbf{R})$ can be embedded in $P_n(\mathbf{R}) \times M$ it follows, according to 3.1, that $\varphi(P_n(\mathbf{R}) \times M, \mathbf{R}^m) \geq \varphi(P_n(\mathbf{R}), \mathbf{R}^m)$. But in the given hypothesis we get that $\varphi(P_n(\mathbf{R}), \mathbf{R}^m) \geq \aleph_0$, because of theorem 2.8, that is we have

$$\varphi(P_n(\mathbf{R}) \times M, S^m) = \varphi(P_n(\mathbf{R}) \times M, \mathbf{R}^m) \geq \varphi(P_n(\mathbf{R}), \mathbf{R}^m) \geq \aleph_0.$$

(*ii*) Follows easily from the equality $\varphi(P_n(\mathbf{R}) \times M, S^m) = \varphi(P_n(\mathbf{R}) \times M, \mathbf{R}^m)$ and from the Whitney's embedding theorem.□

Theorem 3.4. *If* $m \geq 3, n \geq 2$ *are natural numbers and M is a compact connected differentiable manifold such that* $n + \dim M = m$, *then*

(*i*) $\varphi(P_n(\mathbf{R}) \times M, T^a \times S^{m-a}) \geq \aleph_0$; $\forall\, a \in \{1, \ldots, m-2\}$
(*ii*) $\varphi(P_n(\mathbf{R}) \times M, T^m) \geq \aleph_0$
(*iii*) $\varphi(P_n(\mathbf{R}) \times M, S^m) \geq \aleph_0$ *if* $m \geq 4$.

Proof. (*i*) Because $\pi(T^a \times S^{m-a}) \simeq \pi(T^a) \times \pi(S^{m-a}) \simeq \underbrace{(\mathbf{Z} \times \cdots \times \mathbf{Z})}_{\text{a times}}$, it follows that $\pi(T^a \times S^{m-a})$ has no subgroup isomorphic with $\pi(P_n(\mathbf{R}) \times M) \simeq \pi(P_n(\mathbf{R})) \times \pi(M) \simeq \mathbf{Z}_2 \times \pi(M)$. Therefore, according to theorem 2.1 (*i*), it follows that $\varphi(P_n(\mathbf{R}) \times M, T^a \times S^{m-a}) \geq \aleph_0$. The inequality $\varphi(P_n(\mathbf{R}) \times M, T^m) \geq \aleph_0$ can be proved in the same way.

(*iii*) Because $\pi_n(P_n(\mathbf{R}) \times M) \simeq \pi_n(P_n(\mathbf{R})) \times \pi_n(M) \simeq \pi_n(S^n) \times \pi_n(M) \simeq \mathbf{Z} \times \pi_n(M)$ and $\pi_n(S^m) = 0$ it follows that $\pi_n(P_n(\mathbf{R}) \times M) \not\simeq \pi_n(S^m)$. Therefore, according to theorem 2.1 (*ii*), it follows that $\varphi(P_n(\mathbf{R}) \times M, S^m) \geq \aleph_0$.□

Theorem 3.5. *If m,n are natural numbers and M a compact connected differential manifold such that* $n + \dim M \geq m \geq 2$ *and* $\pi(M)$ *is a torsion group, then*

$$\varphi(P_n(\mathbf{R}) \times M, T^a \times S^{m-a}) \geq \aleph_0, \ \forall\, a \in \{1, \ldots, m-1\}.$$

Proof. Because $\pi(P_n(\mathbf{R}) \simeq \mathbf{Z}_2$ and $\pi(M)$ are torsion groups, it follows that $\pi\big(P_n(\mathbf{R}) \times M\big) \simeq \pi(P_n(\mathbf{R}) \times \pi(M) \simeq \mathbf{Z}_2 \times \pi(M)$ is a torsion group too. Because $\pi(T^a \times S^{m-a}) \simeq \pi(T^a) \times \pi(S^{m-a}) \simeq \underbrace{(\mathbf{Z} \times \cdots \times \mathbf{Z})}_{\text{a times}} \times \pi(S^{m-a})$ is a free torsion group, it follows that $\mathrm{Hom}\Big(\pi\big(P_n(\mathbf{R}) \times M\big), \pi(T^a \times S^{m-a})\Big) = 0$, that is

$$\varphi_{alg}\Big(\pi\big(P_n(\mathbf{R}) \times M\big), \pi(T^a \times S^{m-a})\Big) \geq \aleph_0,$$

which means, according to theorem 2.3, that

$$\varphi\big(P_n(\mathbf{R}) \times M, T^a \times S^{m-a}\big) \geq \aleph_0 . \square$$

Theorem 3.6. *If m,n are natural numbers and M a compact connected differential manifold such that $n + \dim M \geq m \geq 2$ and $\pi(M)$ is a free abelian group with rank$\pi(M) < m - 1$, then*

$$\varphi(P_n(\mathbf{R}) \times M, T^a \times S^{m-a}) \geq \aleph_0 \; \forall \, a \in \{\mathrm{rank}\, \pi(M) + 1, \ldots, m-1\}.$$

Proof. Because $\pi\big(P_n(\mathbf{R}) \times M\big) \simeq \mathbf{Z}_2 \times \pi(M)$ it follows that

$$\frac{\pi\big(P_n(\mathbf{R}) \times M\big)}{t\Big(\pi\big(P_n(\mathbf{R}) \times M\big)\Big)} \simeq \pi(M).$$

Therefore rank$\frac{\pi\big(P_n(\mathbf{R}) \times M\big)}{t\Big(\pi\big(P_n(\mathbf{R}) \times M\big)\Big)}$ =rank $\pi(M) < a$ =rank $\pi(T^a \times S^{m-a})$. Using proposition 2.2, it follows that $\varphi_{alg}(\pi\big(P_n(\mathbf{R}) \times M\big), \pi(T^a \times S^{m-a})) \geq \aleph_0$, that is, according to theorem 2.3, one can conclude that $\varphi(P_n(\mathbf{R}) \times M, T^a \times S^{m-a}) \geq \aleph_0 . \square$

4. Applications

Example 4.1. *Let $n_1, \ldots, n_p$ be natural numbers such that $n_i + 1, n_i + 2$ are not powers of 2, for some $i \in \{1, \ldots, p\}$.*

(i) If $n_1 + \ldots + n_p < m \leq 2^{[\log_2 n_i]+1} - 2$, then

$$\varphi(P_{n_1}(\mathbf{R}) \times \ldots \times P_{n_p}(\mathbf{R}), T^a \times \mathbf{R}^{m-a}) \geq \aleph_0 \; (\forall) \, a \in \{1, \ldots, m-1\} \textit{ and}$$

$$\varphi(P_{n_1}(\mathbf{R}) \times \ldots \times P_{n_p}(\mathbf{R}), S^m) = \varphi(P_{n_1}(\mathbf{R}) \times \ldots \times P_{n_p}(\mathbf{R}), \mathbf{R}^m) \geq \aleph_0$$

(ii) If $m \geq 2(n_1 + \ldots + n_p)$, then

$$\varphi(P_{n_1}(\mathbf{R}) \times \ldots \times P_{n_p}(\mathbf{R}), T^a \times \mathbf{R}^{m-a}) = 0 \; \forall \, a \in \{1, \ldots, m-1\},$$

and $\varphi(P_{n_1}(\mathbf{R}) \times \ldots \times P_{n_p}(\mathbf{R}), S^m) = \varphi(P_{n_1}(\mathbf{R}) \times \ldots \times P_{n_p}(\mathbf{R}), \mathbf{R}^m) = 0$

Proof. It is enough to take in the theorems 3.2, 3.3

$$M = P_{n_1}(\mathbf{R}) \times \ldots \times P_{n_{i-1}}(\mathbf{R}) \times P_{n_{i+1}}(\mathbf{R}) \times \ldots \times P_{n_p}(\mathbf{R}).\square$$

Example 4.2. (i) *If* $m, n_1, \ldots, n_p \geq 2$ *are natural numbers such that* $n_1 + \ldots + n_p \geq m \geq 2$*, then* $\varphi(P_{n_1}(\mathbf{R}) \times \ldots \times P_{n_p}(\mathbf{R}), T^a \times S^{m-a}) \geq \aleph_0,\ (\forall)\, a \in \{1, \ldots, m-1\}$.

(ii) *If* $a, b, m, n_1 \ldots n_p \geq 2$ *are natural numbers such that* $a < b$ *and* $a + n_1 + \ldots + n_p \geq m \geq 2$*, then* $\varphi(P_{n_1}(\mathbf{R}) \times \ldots \times P_{n_p}(\mathbf{R}) \times T^a, T^b \times S^{m-b}) \geq \aleph_0$.

Proof. (i) It is enough to take in the theorem 3.5 $M = P_{n_2}(\mathbf{R}) \times \ldots \times P_{n_p}(\mathbf{R})$.

(ii) It is enough to take in the theorem 3.6 $M = P_{n_2}(\mathbf{R}) \times \ldots \times P_{n_p}(\mathbf{R}) \times T^b$.$\square$

References

[AnPi] Andrica, D., Pintea, C., *Critical points of vector-valued functions*, Proc. 24^{th} Conf. Geom. Top., Univ. Timişoara.

[Pi1] Pintea, C., *Differentiable mappings with an infinite number of critical points*, Proc. Amer. Math. Soc., Vol. 128, No. 11, 2000, 3435-3444.

[Pi2] Pintea, C., *Continuous mappings with an infinite number of topologically critical points*, Ann. Polonici Math. LXVII.1 (1997).

[Pi3] Pintea, C., *The φ-category of the pairs* $(G_{k,n}, S^m), (P_n(\mathbf{R}), T^a \times \mathbf{R}^{m-a})$ (to be published)

[Pi4] Pintea, C., *A measure of non-immersability of the Grassmann manifolds in some Euclidean spaces*, Proc. Edinburgh Math. Soc. **41**(1998), 197-206.

Faculty of Mathematics and Computer Science,
Babeş-Bolyai University, 3400 Cluj-Napoca, Romania
E-mail address: cpintea@math.ubbcluj.ro

Received: 12.11.2001