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COMMON FIXED POINT THEOREMS FOR MULTIVALUED OPERATORS ON COMPLETE METRIC SPACES

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1. Introduction

The purpose of this paper is to prove a common fixed point theorem for multivalued operators defined on a complete metric space. Then, as consequences, we obtain some generalizations of several results proved in [6] for singlevalued operators.

For other results of this type see [1], [2], [3] and [5]. The metric conditions which appears in Theorem 3.1 generalize some conditions given in [6].

2. Preliminaries

Let X be a nonempty set. We denote:

$$P(X) := \{A \subset X \mid A \neq \varnothing\} \quad \text{and} \quad P_{cl}(X) := \{A \in P(X) \mid A = \bar{A}\}.$$

If (X, d) is a metric space, $B \in P(X)$ and $a \in A$, then

$$D(a, B) := \inf\{d(a, b) \mid b \in B\}.$$

Definition 2.1. If $T : X \multimap X$ is a multivalued operator, then an element $x \in X$ is a ***fixed point of T***, iff $x \in T(x)$.

We denote by $F_T := \{x \in X \mid x \in T(x)\}$ ***the fixed points set of T***.

Definition 2.2. Let $(T_n)_{n\in\mathbb{N}^*}$ be a sequence of multivalued operators $T_n : X \to P(X)$, $(\forall)\ n \in \mathbb{N}^*$. Then we denote by

$$Com(T) := \{x \in X \mid x \in T_n(x), \quad (\forall)\, n \in \mathbb{N}^*\} = \bigcap_{n\in\mathbb{N}^*} F_{T_n}$$

the ***common fixed points*** set of the sequence $(T_n)_{n\in\mathbb{N}^*}$.

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Lemma 2.3. (I.A.Rus [4]). *Let* $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+ \quad (k \in \mathbb{N}^*)$ *be a function and denote by* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, *the mapping given by* $\psi(t) = \varphi(t, t, ..., t), \quad (\forall)\ t \in \mathbb{R}_+$.

Suppose that the following conditions are satisfied:

i) $(r \leqslant s, \quad r, s \in \mathbb{R}_+^k) \quad \Rightarrow \quad \varphi(r) \leqslant \varphi(s)$;

ii) φ *is upper semi-continuous;*

iii) $\psi(t) < t$, *for each* $t > 0$.

Then $\lim\limits_{n\to\infty} \psi^n(t) = 0$, *for each* $t \geqslant 0$.

In [6], T.Veerapandi and S.A.Kumar gave the following result:

Theorem 2.4. *Let* X *be a Hilbert space,* $Y \in P_{cl}(X)$ *and* $T_n : Y \to Y$, *for* $n \in \mathbb{N}$, *be a sequence of mappings.*

We suppose that at least one of the following conditions is satisfied:

i) there exist real numbers $a, b, c,$ *satisfying* $0 \leqslant a, b, c < 1$ *and* $a + 2b + 2c < 1$ *such that for each* $x, y \in Y$ *and* $x \neq y$,

$$\|T_i(x) - T_j(y)\|^2 \leqslant a \cdot \|x - y\|^2 + b\left(\|x - T_i(x)\|^2 + \|y - T_j(y)\|^2\right) + \\ + \frac{c}{2}\left(\|x - T_j(y)\|^2 + \|y - T_i(x)\|^2\right), \text{ for } i, j;$$

ii) there exist a real number h *satisfying* $0 \leqslant h < 1$ *such that for all* $x, y \in Y$ *and* $x \neq y$,

$$\|T_i(x) - T_j(y)\|^2 \leqslant h \cdot \max\left\{\|x - y\|^2, \frac{1}{2}\left(\|x - T_i(x)\|^2 + \|y - T_j(y)\|^2\right), \\ \frac{1}{4}\left(\|x - T_j(y)\|^2 + \|y - T_i(x)\|^2\right)\right\}, \text{ for } i, j.$$

Then, $(T_n)_{n\in\mathbb{N}^*}$ *has a unique common fixed point.*

3. The main results

The first result of this section improve and generalize Theorem 2.4 in the multivoque case.

Theorem 3.1. *Let (X, d) be a complete metric space and* $S, T : X \to P_{cl}(X)$ *multivalued operators.*

We suppose that there exists a function $\varphi : \mathbb{R}_+^3 \to \mathbb{R}_+$ *such that:*

i) $(r \leqslant s, \quad r, s \in \mathbb{R}_+^3) \quad \Rightarrow \quad \varphi(r) \leqslant \varphi(s)$;

ii) $\varphi(t, t, t) < t$ *for each* $t > 0$;

iii) φ *is continuous;*

iv) for each $x \in X$, *any* $u_x \in S(x)$ *and for all* $y \in X$, *there exists* $u_y \in T(y)$ *so that we have*

$$d^2(u_x,\, u_y) \leqslant \varphi\left(d^2(x,y)\,,\quad \frac{d^2(x,u_x)+d^2(y,u_y)}{2}\,,\quad \frac{d^2(x,u_y)+d^2(y,u_x)}{4}\right).$$

In these conditions, $F_S = F_T = \{x^*\}$.

Proof. Let $x_0 \in X$ arbitrarily. Then we can construct a sequence $(x_n) \subset X$ such that

$$\begin{cases} x_{2n+1} \in S(x_{2n}) \\ x_{2n+2} \in T(x_{2n+1}) \end{cases} \quad (\forall)\, n \in \mathbb{N}.$$

Denote by $d_n := d(x_n\,,\, x_{n+1})$, $n \in \mathbb{N}$. We have several steps in our proof.

Step I. Let us prove that the sequence (d_n) is monotone decreasing. Indeed, we have successively:

$$d^2_{2n+1} = d^2(x_{2n+1}, x_{2n+2}) \leqslant$$
$$\leqslant \varphi\Big(d^2(x_{2n}, x_{2n+1}), \frac{d^2(x_{2n}, x_{2n+1}) + d^2(x_{2n+1}, x_{2n+2})}{2},$$
$$\frac{d^2(x_{2n}, x_{2n+2}) + d^2(x_{2n+1}, x_{2n+1})}{4}\Big) \leqslant$$
$$\leqslant \varphi\left(d^2_{2n}\,,\ \frac{d^2_{2n}+d^2_{2n+1}}{2}\,,\ \frac{(d_{2n}+d_{2n+1})^2}{4}\right) < \max\left\{d^2_{2n}\,,\ \frac{d^2_{2n}+d^2_{2n+1}}{2}\right\} = d^2_{2n},$$

from where it follows $d_{2n+1} < d_{2n}$. By an analogous method we have $d_{2n+2} < d_{2n+1}$.

Step II. We prove that $\lim\limits_{n\to\infty} d_n = 0$.

For this purpose, let us define $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, by $\psi(t) = \varphi(t,t,t)$. Obviously, ψ is monotone increasing and $\psi(t) < t$, $(\forall)\, t > 0$.

By induction, we can prove that $d_n^2 \leqslant \psi^n(d_0^2)$, $(\forall)\, n \geqslant 1$.

Indeed, we have

$$d_1^2 \leqslant \varphi\left(d_0^2\,,\ \frac{d_1^2+d_0^2}{2}\,,\ \frac{(d_0+d_1)^2}{4}\right) \leqslant \varphi(d_0^2,\ d_0^2,\ d_0^2) = \psi(d_0^2).$$

If inequality $d^2_{2n} \leqslant \psi^{2n}(d_0^2)$ is true, then we get successively:

$$d^2_{2n+1} \leqslant \varphi\left(d^2_{2n}\,,\ \frac{d^2_{2n}+d^2_{2n+1}}{2}\,,\ \frac{(d_{2n}+d_{2n+1})^2}{4}\right) \leqslant \varphi(d^2_{2n}, d^2_{2n}, d^2_{2n}) = \psi(d^2_{2n}) \leqslant$$
$$\leq \psi(\psi^{2n}(d_0^2)) = \psi^{2n+1}(d_0^2).$$

By passing to limit as $n \to \infty$, if $d_0 > 0$ it follows

$$\lim_{n\to\infty} d_n^2 \leqslant \lim_{n\to\infty} \psi^n(d_0^2) = 0, \quad \text{and hence} \quad \lim_{n\to\infty} d_n = 0.$$

For $d_0 = 0$, the sequence (d_n) being decreasing it is obviously that $\lim_{n\to\infty} d_n = 0$.

Step III. We'll prove that the sequence (x_n) is Cauchy in X, i.e. for each $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for each $m, n \geqslant k$, $d(x_m, x_n) < \varepsilon$.

Suppose, by contradiction, that (x_{2n}) is not Cauchy sequence. Then, there exists $\varepsilon > 0$ such that for each $2k \in \mathbb{N}$ there exist $2m_k, 2n_k \in \mathbb{N}$, $2m_k > 2n_k \geqslant 2k$, with the property $d(x_{2m_k}, x_{2n_k}) > \varepsilon$.

In what follows, let us suppose the numbers $2m(k)$ and $2n(k)$ as follows:

$$2m(k) := \inf\{2m_k \in \mathbb{N} \mid 2m_k > 2n_k \geqslant 2k,\ d(x_{2n_k}, x_{2m_k-2}) \leqslant \varepsilon,\ d(x_{2n_k}, x_{2m_k}) > \varepsilon\}$$

and $2n(k) := 2n_k$. Then, $(\forall)\, 2k \in \mathbb{N}$ we have:

$$\varepsilon < d(x_{2n(k)}, x_{2m(k)}) \leqslant d(x_{2n(k)}, x_{2m(k)-2}) + d(x_{2m(k)-2}, x_{2m(k)-1}) +$$
$$+ d(x_{2m(k)-1}, x_{2m(k)}).$$

Using step II, we deduce that

$$\lim_{k\to\infty} d(x_{2n(k)}, x_{2m(k)}) = \varepsilon. \tag{1}$$

From the triangle inquality, we get:

$$|d(x_{2n(k)}, x_{2m(k)-1}) - d(x_{2n(k)}, x_{2m(k)})| \leqslant d(x_{2m(k)-1}, x_{2m(k)})$$

and

$$|d(x_{2n(k)+1}, x_{2m(k)-1}) - d(x_{2n(k)}, x_{2m(k)})| \leqslant d(x_{2m(k)-1}, x_{2m(k)}) + d(x_{2n(k)}, x_{2n(k)+1}).$$

Using again step III and the relation (1), it follows

$$\begin{cases} \lim_{k\to\infty} d(x_{2n(k)}, x_{2m(k)-1}) = \varepsilon \\ \lim_{k\to\infty} d(x_{2n(k)+1}, x_{2m(k)-1}) = \varepsilon. \end{cases} \tag{2}$$

Then, we have successively:

$$d(x_{2n(k)}, x_{2m(k)}) \leqslant d(x_{2n(k)}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2m(k)}) \leqslant d(x_{2n(k)}, x_{2n(k)+1}) +$$
$$+ \Bigg[\varphi(d^2(x_{2n(k)}, x_{2m(k)-1}), \ \frac{d^2(x_{2n(k)}, x_{2n(k)+1}) + d^2(x_{2m(k)-1}, x_{2m(k)})}{2},$$

$$\left. \frac{d^2(x_{2n(k)}, x_{2m(k)}) + d^2(x_{2m(k)-1}, x_{2n(k)+1})}{4} \right]^{\frac{1}{2}}.$$

Because φ is continuous, passing to the limit as $k \to \infty$, we have:

$$\varepsilon \leqslant \left[\varphi\left(\varepsilon^2, 0, \frac{\varepsilon^2}{2}\right)\right]^{\frac{1}{2}} \leqslant \left[\psi(\varepsilon^2)\right]^{\frac{1}{2}} < \varepsilon, \quad \text{a contradiction.}$$

Step IV. We prove that $F_T \neq \varnothing$.

Because (x_n) is Cauchy sequence in the complete metric space (X, d) we obtain that there exists $x^* \in X$ such that $\lim\limits_{n\to\infty} x_n = x^*$.

From $x_{2n+1} \in S(x_{2n})$ we have that there exists $u_n \in T(x^*)$ such that:

$$d^2(x_{2n+1}, u_n) \leqslant$$

$$\leq \varphi\left(d^2(x_{2n}, x^*)\,, \frac{d^2(x_{2n}, x_{2n+1}) + d^2(x^*, u_n)}{2}\,, \frac{d^2(x_{2n}, u_n) + d^2(x^*, x_{2n+1})}{4}\right) <$$

$$< \max\left\{d^2(x_{2n}, x^*)\,, \frac{d^2(x_{2n}, x_{2n+1}) + d^2(x^*, u_n)}{2}\,, \frac{d^2(x_{2n}, u_n) + d^2(x^*, x_{2n+1})}{4}\right\}$$

$$:= M.$$

Consequently, we have the following situations:

a. Case $M = d^2(x_{2n}, x^*)$. In this case, we have

$$d^2(x_{2n+1}, u_n) \leqslant d^2(x_{2n}, x^*),$$

from where

$$\lim_{n\to\infty} d(x_{2n+1}, u_n) \leqslant \lim_{n\to\infty} d(x_{2n}, x^*) = 0,$$

i.e.

$$\lim_{n\to\infty} d(x_{2n+1}, u_n) = 0.$$

b. Case $M = \dfrac{d^2(x_{2n}, x_{2n+1}) + d^2(x^*, u_n)}{2}$. We deduce successively:

$$d^2(x_{2n+1}, u_n) \leqslant \frac{d^2(x_{2n}, x_{2n+1}) + d^2(x^*, u_n)}{2} \leqslant$$

$$\leq \frac{d^2(x_{2n}, x_{2n+1}) + [d(x^*, x_{2n+1}) + d(x_{2n+1}, u_n)]^2}{2},$$

i.e. $d^2(x_{2n+1}, u_n) - 2\cdot d(x^*, x_{2n+1})\cdot d(x_{2n+1}, u_n) - [d^2(x_{2n}, x_{2n+1}) + d^2(x^*, x_{2n+1})] \leqslant 0$, therefore

$$d(x_{2n+1}, u_n) \leqslant d(x^*, x_{2n+1}) + \sqrt{2 \cdot d^2(x^*, x_{2n+1}) + d^2(x_{2n}, x_{2n+1})}.$$

Passing to the limit in this inequality, as $n \to \infty$, we obtain

$$\lim_{n\to\infty} d(x_{2n+1}, u_n) = 0.$$

c. Case $M = \dfrac{d^2(x_{2n}, u_n) + d^2(x^*, x_{2n+1})}{4}$. In this case, from the inequality

$$d^2(x_{2n+1}, u_n) \leqslant \frac{d^2(x_{2n}, u_n) + d^2(x^*, x_{2n+1})}{4},$$

we have, again,

$$\lim_{n\to\infty} d(x_{2n+1}, u_n) = 0.$$

Passing to the limit, as $n \to \infty$, in inequality

$$d(x^*, u_n) \leqslant d(x^*, x_{2n+1}) + d(x_{2n+1}, u_n),$$

on the basis of the limit $\lim_{n\to\infty} d(x_{2n+1}, u_n) = 0$, we obtain $d(x^*, u_n) \to 0$ as $n \to 0$.

Since $u_n \in T(x^*)$, $(\forall)\, n \in \mathbb{N}$ and $T(x^*)$ is a closed set, it follows that $x^* \in T(x^*)$, i.e. $x^* \in F_T$.

Step V. We'll obtain, now, the conclusion of our theorem. We first prove that $F_S \subset F_T$.

Let $x^* \in F_S$. From $x^* \in S(x^*)$ we have that there exists $u \in T(x^*)$ such that

$$d^2(x^*, u) \leqslant \varphi\left(d^2(x^*, x^*)\,,\ \frac{d^2(x^*, x^*) + d^2(x^*, u)}{2}\,,\ \frac{d^2(x^*, u) + d^2(x^*, x^*)}{4}\right).$$

If we suppose that $d(x^*, u) > 0$, then we obtain

$$d^2(x^*, u) \leqslant \varphi\left(0,\ \frac{d^2(x^*, u)}{2}\,,\ \frac{d^2(x^*, u)}{4}\right) < \frac{d^2(x^*, u)}{2},$$

a contradiction. Thus, $d(x^*, u) = 0$, which means that $u = x^*$. It follows that $x^* \in T(x^*)$ and so $F_S \subset F_T$.

We shall prove now the equality $F_S = F_T$ betwen the fixed points set for S and T.

If we assume that there exists $y^* \in F_T$ such that $y^* \neq x^* \in F_S$, then we have

$$d^2(x^*, y^*) \leqslant \varphi\left(d^2(x^*, y^*)\,,\ \frac{d^2(x^*, x^*) + d^2(y^*, y^*)}{2}\,,\ \frac{d^2(x^*, y^*) + d^2(y^*, x^*)}{4}\right) =$$

$$= \varphi\left(d^2(x^*, y^*)\,,\ 0\,,\ \frac{d^2(x^*, y^*)}{2}\right) \leqslant \psi(d^2(x^*, y^*)) < d^2(x^*, y^*),$$

a contradiction, proving the fact that $F_S = F_T \in P(X)$.

In fact, we have obtained, even more, namely that $F_S = F_T = \{x^*\}$. □

Corollary 3.2. *Let (X, d) be a complete metric space and $S, T : X \to P_{cl}(X)$ multivalued operators .*

We suppose that there exist $a, b, c \in \mathbb{R}_+$, $a + 2b + 2c < 1$, such that for each $x \in X$, each $u_x \in S(x)$ and for all $y \in X$, there exists $u_y \in T(y)$ so that we have

$$d^2(u_x, u_y) \leqslant a \cdot d^2(x, y) + b \cdot [d^2(x, u_x) + d^2(y, u_y)] + \frac{c}{2} \cdot [d^2(x, u_y) + d^2(y, u_x)].$$

Then, $F_S = F_T = \{x^*\}$.

Proof. Applying Theorem 3.1 for the function $\varphi : \mathbb{R}^3_+ \to \mathbb{R}_+$, $\varphi(t_1, t_2, t_3) = at_1 + 2bt_2 + 2ct_3$, which satisfies the conditions i), ii) and iii) of this theorem, we obtain the conclusion. □

Remark 3.3. If T and S are singlevalued operators, then Corollary 3.2 is Theorem 3 from [6].

Corollary 3.4. *Let (X, d) be a complete metric space and $S, T : X \to P_{cl}(X)$ multivalued operators.*

We suppose that there exists $h \in]0, 1[$ such that for each $x \in X$, any $u_x \in S(x)$ and for all $y \in X$, there exists $u_y \in T(y)$ so that we have

$$d^2(u_x, u_y) \leqslant h \cdot \max\left\{d^2(x, y)\,,\ \frac{d^2(x, u_x) + d^2(y, u_y)}{2}\,,\ \frac{d^2(x, u_y) + d^2(y, u_x)}{4}\right\}.$$

In these conditions, $F_S = F_T = \{x^*\}$.

Proof. We apply Theorem 3.1 for the function $\varphi : \mathbb{R}^3_+ \to \mathbb{R}_+$, $\varphi(t_1, t_2, t_3) = h \cdot \max\{t_1, t_2, t_3\}$, which satisfies the conditions i), ii) and iii) of this theorem. □

Remark 3.5. Corollary 3.4 is a generalization for multivalued operators of Theorem 4 from [6], theorem proved for singlevalued operators in Hilbert spaces.

Remark 3.6. Let (X, d) be a complete metric space and $(T_n)_{n\in\mathbb{N}}$ be a sequence of multivalued operators $T_n : X \to P_{cl}(X)$, $(\forall)\, n \in \mathbb{N}$.

If each pair of multivalued operators (T_0, T_n), for $n \in \mathbb{N}^*$, satisfies similar conditions as in Theorem 3.1, then $F_{T_n} = F_{T_0} = \{x^*\}$, for all $n \in \mathbb{N}^*$.

We next give a generalization of Theorem 1 of N.Negoescu [2].

Theorem 3.7. *Let (X, d) be a compact metric space,* $S, T : X \to P_{cl}(X)$ *and* $\varphi : \mathbb{R}^3_+ \to \mathbb{R}_+$. *Suppose that the following conditions are satisfies:*

i) $(r \leqslant s;\ r, s \in \mathbb{R}^3_+) \quad \Rightarrow \quad \varphi(r) \leqslant \varphi(s)$;

ii) $\varphi(t, t, t) < t, \quad (\forall)\, t > 0$;

iii) *S or T be continuous;*

iv) $d^2(u_x, u_y) < \varphi\Big(d^2(x, y)\,,\ d(x, u_x)\cdot d(y, u_y)\,,\ d(x, u_y)\cdot d(y, u_x)\Big)$, *for all* $x, y \in X,\ x \neq y$ *and for all* $(u_x, u_y) \in S(x) \times T(x)$.

In these conditions:

a. *S or T has a strict fixed point;*

b. *if both S and T have such fixed points, then the pair (S, T) has a common fixed point.*

Proof. **a.** Let S be continuous and we consider the function $f(x) := D(x,\, S(x))$. Because f is continuous on X, it follows that f takes its minimum value, i.e. there exists $x_0 \in X$ such that $f(x_0) = \inf\{f(x) \mid x \in X\}$.

We prove that x_0 is a fixed point of S or some $x_1 \in S(x_0)$ is a fixed point of T.

Indeed, we choose:

$x_1 \in S(x_0)$ be such that $d(x_0, x_1) = D(x_0, S(x_0))$;

$x_2 \in T(x_1)$ be such that $d(x_1, x_2) = D(x_1, T(x_1))$;

$x_3 \in S(x_2)$ be such that $d(x_2, x_3) = D(x_2, S(x_2))$.

We shall prove that $D(x_0, S(x_0)) = 0$ or $D(x_1, T(x_1)) = 0$, i.e. $x_0 \in S(x_0)$ or $x_1 \in T(x_1)$. We suppose that $D(x_0, S(x_0)) > 0$ and $D(x_1, T(x_1)) > 0$. Hence, using the inequality iv), we have:

$$d^2(x_1, x_2) < \varphi\Big(d^2(x_0, x_1)\,,\ d(x_0, x_1)\cdot d(x_1, x_2)\,,\ d(x_0, x_2)\cdot d(x_1, x_1)\Big) \leqslant$$

$$\leqslant \max\Big\{d^2(x_0, x_1)\,,\ d(x_0, x_1)\cdot d(x_1, x_2)\Big\} := M.$$

Consequently, we distinguish the following situations:

I. Case $M = d^2(x_0, x_1)$**.** In this case, we deduce $d(x_1, x_2) < d(x_0, x_1)$.

II. Case $M = d(x_0, x_1)\cdot d(x_1, x_2)$**.** In this case, we have $d^2(x_1, x_2) < d(x_0, x_1)\cdot d(x_1, x_2)$. Since $d(x_1, x_2) = D(x_1, T(x_1)) > 0$, it follows that $d(x_1, x_2) <$

$d(x_0, x_1)$. Now,

$$d^2(x_3, x_2) < \varphi\bigg(d^2(x_2, x_1)\,,\ d(x_2, x_3)\cdot d(x_1, x_2)\,,\ d(x_2, x_2)\cdot d(x_1, x_3)\bigg) \leqslant$$

$$\leqslant \max\bigg\{d^2(x_1, x_2)\,,\ d(x_2, x_3))\cdot d(x_1, x_2)\bigg\}.$$

Analogously, it follows that $d^2(x_2, x_3) < d^2(x_1, x_2)$ or $d^2(x_2, x_3) < d(x_2, x_3)\cdot d(x_1, x_2)$.

In the second situations, if $d(x_2, x_3) = 0$, we obtain a contradiction. Thus, it follows that $d(x_2, x_3) < d(x_1, x_2)$.

Similarly, we deduce successively:

$$D(x_2, S(x_2)) = d(x_2, x_3) < d(x_1, x_2) < d(x_0, x_1) = f(x_0),$$

which contradict the minimality of $f(x_0)$. Therefore, $D(x_0, S(x_0)) = 0$ or $D(x_1, T(x_1)) = 0$. So, $x_0 \in S(x_0)$ or $x_1 \in T(x_1)$.

b. We assume that there exist $u \in S(u)$ and $v \in T(v)$, such that $u \neq v$. Then, using the hypothesis iv) we get, again, a contradiction:

$$d^2(u, v) < \varphi\bigg(d^2(u, v)\,,\ d(u, u)\cdot d(v, v)\,,\ d^2(u, v)\bigg) \leqslant d^2(u, v).$$

So, u=v, meaning that u is a common fixed point of S and T. □

Remark 3.8. If $\varphi : \mathbb{R}^3_+ \to \mathbb{R}_+$, $\varphi(t_1, t_2, t_3) = \max\{t_1, t_2, t_3\}$, from Theorem 3.7, we get a result of Negoescu [2, Theorem 1].

Remark 3.9. We note that Theorem 3.7 is true for $S = T : X \to P_{cl}(X)$.

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