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RELATION BETWEEN THE PALAIS-SMALE CONDITION AND COERCIVENESS FOR MULTIVALUED MAPPINGS

MEZEI ILDIKÓ ILONA

Abstract. The aim of this paper is to study the coerciveness property of a class of multivalued mappings satisfying the Palais-Smale condition.

1. **Introduction**

Many papers has been devoted to show that the Palais-Smale condition implies the coerciveness. In the differentiable case this property is studied by L. Caklovici, S.Li, and M. Willem [2], for the locally Lipschitz functionals by Cs. Varga and V. Varga [11]. For the class of functions introduced by A. Szulkin [10], which is lower semicontinuous, this property has been proved by D. Goeleven in the paper [7]. For continuous functionals this result is proved by Fang [6]. These results are generalized by J.-N. Corvellec, see [4].

In a recent paper D. Motreanu and V.V. Motreanu [8] studied this problem for a class of functional of type $\Phi + \gamma$, where Φ is a locally Lipschitz function and γ is a proper, convex, lower semicontinuous functional.

In this paper we study the coerciveness of the function $\gamma + \sigma$, where σ is a locally Lipschitz function and γ is a convex lower semicontinuous function. The main tool used in the proof the coerciveness property is the classical Ekeland's variational principle [5].

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2. Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space and let $A : X \rightsquigarrow X$ be a multivalued map with $A(x) \neq \emptyset$, $\forall x \in X$, i.e $DomA = X$. Let X^* be the dual of X.

Definition 2.1 [1] $A : X \rightsquigarrow X$ is *Lipschitz around* $x \in X$ if there exists a positive constant l and a neighborhood U of x such that

$$\forall x_1, x_2 \in U,\ \|y_1 - y_2\| \leq l\|x_1 - x_2\|, \quad \forall y_1 \in A(x_1),\ y_2 \in A(x_2).$$

If A is Lipschitz around all $x \in X$, we say that A is *locally Lipschitz.*

Definition 2.2[9] *The generalized directional derivative* of the locally Lipschitz function $f : X \to \mathbb{R}$ at the point $x_0 \in X$ in the direction $h \in X$ is defined by

$$f^0(x_0, h) = \limsup_{x \to x_0\ t \searrow 0} \frac{f(x + th) - f(x)}{t}.$$

Let $p \in X^*$ such that $\|p\|_* < \infty$, where $\|p\|_* = sup\{\langle p, x\rangle : \|x\| \leq 1, x \in X\}$.

Lemma 2.1 *If $A : X \rightsquigarrow X$ is locally Lipschitz, then the function $x \mapsto \sigma(A(x), p)$ is locally Lipschitz, where*

$$\sigma(A(x), p) = sup\{\langle p, y\rangle : y \in A(x)\}, p \in X^*.$$

Proof. We consider an arbitrary $x_0 \in X$. Since A is locally Lipschitz, there exist $l > 0$ and an U neighborhood of x_0 such that:

$$\forall x_1, x_2 \in U, \forall y_1 \in A(x_1), y_2 \in A(x_2) : \|y_1 - y_2\| \leq l\|x_1 - x_2\|.$$

We can suppose that $\sigma(Ax_1, p) \geq \sigma(Ax_2, p)$. It's easy to verify that

$$0 \leq \sigma(Ax_1, p) - \sigma(Ax_2, p) \leq \sup_{y_1 \in Ax_1, y_2 \in Ax_2} \langle p, y_1 - y_2\rangle.$$

But

$$\sup_{y_i \in Ax_i} \langle p, y_1 - y_2\rangle = \sup_{y_i \in Ax_i} \langle p, \frac{y_1 - y_2}{\|y_1 - y_2\|}\|y_1 - y_2\|\rangle =$$
$$= \sup_{y_i \in Ax_i} \langle p, \frac{y_1 - y_2}{\|y_1 - y_2\|}\rangle \cdot \|y_1 - y_2\| \leq \|p\|_* \cdot l \cdot \|x_1 - x_2\|,$$

providing that $y_1 \neq y_2$ .The case $y_1 = y_2$ is trivial.

Therefore $x \mapsto \sigma(Ax, p)$ is locally Lipschitz. □

We consider an appropriate class of function as [9, chapter3].

Let $J : X \to R$ be a function given by

$$(H) \qquad J(x) = \psi(x) + \sigma(A(x), p),$$

where $\psi : X \to R$ is a convex lower semicontinuous function, $A : X \rightsquigarrow X$ is a locally Lipschitz multivalued map and $p \in X^*$.

Definition 2.3 A point $u \in X$ is said to be *critical point of J for $p \in X^*$* if it satisfies the following variational inequality

$$\psi(v) - \psi(u) + (\sigma(A(\cdot), p))^0(u, v - u) \geq 0, \ \ \forall v \in X.$$

Definition 2.4 The function J satisfies the *Palais-Smale condition at level c* (briefly $(\mathrm{PS})_c$) if for each sequence $\{u_n\} \subset X$ such that $J(u_n) \to c$ and $\psi(v) - \psi(u_n) - (\sigma(A(\cdot), p))^0(u_n, v - u_n) \geq -\varepsilon_n \|v - u_n\|$, $\forall v \in X$, where $\varepsilon_n \to 0$, $\{u_n\}$ contains a convergent subsequence.

Definition 2.5 We say that J is *coercive*, if for $\|u\| \to \infty$ we have $J(u) \to \infty$.

As we said above our main tool is the Ekeland's principle, which we recall now.

Theorem 2.1 *Let X be a complete metric space and let $f : X \to (-\infty, \infty]$ be a lower semicontiuous function such that $inf_X f \in \mathbb{R}$. Let $\varepsilon > 0$ and $u \in X$ be given such that $f(u) \leq inf_X f + \varepsilon$. Then for every $\lambda > 0$, there exists an element $v \in X$, such that*

i) $f(v) < f(u)$;

ii) $f(v) < f(w) + \frac{\varepsilon}{\lambda} \cdot d(v, w)$, *for every* $w \neq v$;

iii) $d(u, v) \leq \lambda$.

3. Main result

Theorem 3.1 *Let X be a Banach space,J a bounded bellow function satisfying* (H) *and $p \in X^*$ such that $\|p\|_* < \infty$. Define*

$$c := \liminf_{\|u\| \to \infty} J(u).$$

Then, if $c \in \mathbb{R}$, there exists a sequence $\{v_n\} \subset X$ such that:

(i) $\|v_n\| \to \infty$;

(ii) $J(v_n) \to c$;

(iii) $\psi(v) - \psi(v_n) + (\sigma(A(\cdot), p))^0(v_n, v - v_n) \geq -\varepsilon_n \cdot \|v - v_n\|$, *where* $\varepsilon_n \to 0,\ \forall v \in X$.

Proof. From the definition of c there exists a sequence u_n such that $J(u_n) \leq c + \frac{1}{n}$ and $\|u_n\| \geq 2n$, for $n \in N \setminus \{0\}$ sufficiently large. Evidently J is lower semicontinuous and so we can apply the Theorem 2.1, with $f = J$, $\varepsilon = c + \frac{1}{n} - \inf_X J$ and $\lambda = n$.

Thus there exists $v_n \in X$ such that:

$$J(v_n) \leq J(u_n) \leq c + \frac{1}{n}; \tag{1}$$

$$J(w) > J(v_n) - \frac{1}{n}\big(c + \frac{1}{n} - inf_X J\big)\|v_n - w\|,\ \ \forall w \neq v_n;$$

$$\|u_n - v_n\| \leq n. \tag{2}$$

Thus, for each $w \in X$ we have

$$J(w) - J(v_n) \geq -\frac{1}{n}\big(c + \frac{1}{n} - inf_X J\big)\|w - v_n\|.$$

Let $w = (1-t)v_n + tv$, where v is fixed in X and $t \in [0, 1]$. Replacing w in the last inequality we obtain

$$\psi(v_n + t(v - v_n)) - \psi(v_n) + \sigma(A((1-t)v_n + tv), p) - \sigma(A(v_n), p) \geq -\varepsilon_n t\|v - v_n\|,$$

where $\varepsilon_n = \big(c + \frac{1}{n} - \inf_X J\big)\frac{1}{n}$.

Since ψ is convex, we have

$$t(\psi(v) - \psi(v_n)) + \sigma(A((1-t)v_n + tv), p) - \sigma(A(v_n), p) \geq -\varepsilon_n t\|v - v_n\|.$$

Dividing this relation by t we get

$$\psi(v) - \psi(v_n) + \frac{1}{t}\Big[\sigma(A(v_n + t(v - v_n)), p) - \sigma(A(v_n), p)\Big] \geq -\varepsilon_n\|v - v_n\|. \tag{3}$$

Taking the limit as $t \searrow 0$ and using that

$$\sigma(A(\cdot,p))^0(v_n, v - v_n) = \limsup_{w_n \to v_n \ t \searrow 0} \frac{\sigma(A(w_n + t(v - v_n)), p) - \sigma(A(w_n), p)}{t} \geq$$

$$\geq \lim_{t \searrow 0} \frac{\sigma(A(v_n + t(v - v_n)), p) - \sigma(A(v_n), p)}{t}$$

we obtain

$$\psi(v) - \psi(v_n) + (\sigma(A(\cdot), p))^0 (v_n, v - v_n) \geq -\varepsilon_n \|v - v_n\|, \ \ \varepsilon_n \to 0,$$

$\forall v \in X$ i.e. exactly the (iii).

From (2) and (1) we have $\|v_n\| \geq \|u_n\| - \|u_n - v_n\| \geq 2n - n = n$, and $J(v_n) \to c$ respectively thus we have constructed a sequence such that (i), (ii) and (iii) are satisfied. □

Corollary 3.1 *Let X be a Banach space and let $J : X \to \mathbb{R}$ be a function of the form $J(x) = \psi(x) + \sigma(Ax, p)$, with $\|p\|_* < \infty$ satisfying* (H) *and the* (PS) *condition. If J is bounded bellow, then J is coercive.*

Proof. We proceed by contradiction. Assume that

$$c = \liminf_{\|u\| \to \infty} J(u) \in \mathbb{R}.$$

Then by the main theorem, there exists a sequence v_n such that $\|v_n\| \to \infty$, $J(v_n) \to c$ and $\psi(v) - \psi(v_n) + (\sigma(A(\cdot), p))^0 (v_n, v - v_n) \geq -\varepsilon_n \|v - v_n\|, \ \forall v \in X$, where $\varepsilon_n \to 0$. Since J satisfies the (PS) condition, we can choose a convergent subsequence of $\{v_n\}$, which is in contradiction with $\|v_n\| \to \infty$. □

Remark 3.1 The Corollary 3.1 generalize some results from the papers [2], [11], [7] and [8].

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FACULTY OF MATHEMATICS AND COMPUTER SCIENCE,
BABEŞ-BOLYAI UNIVERSITY, 3400 CLUJ-NAPOCA, ROMANIA
E-mail address: ikulcsar@math.ubbcluj.ro

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