STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume XLVII, Number 1, March 2002

CONTINUITY AND SUPERSTABILITY OF JORDAN MAPPINGS

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Abstract. We show that every strong approximate one-to-one Jordan functional on an algebra is a Jordan functional and every approximate one-to-one Jordan functional on a Banach algebra is continuous.

1. Introduction

A linear mapping f from a normed algebra A into a normed algebra B is an ε-homomorphism if for every a, b in A

$$\|f(ab) - f(a)f(b)\| \leq \varepsilon \|a\| \|b\| .$$

In [7, Proposition 5.5], Jarosz proved that every ε-homomorphism from a Banach algebra into a continuous function space $C(S)$ is necessarily continuous, where S is a compact Hausdorff space. A Jordan functional on a Banach algebra A is a nonzero linear functional ϕ such that $\phi(a^2) = \phi(a)^2$ for every a in A. Every Jordan functional ϕ on A is multiplicative [2]. We are concerned with linear mappings f on Banach algebras which are approximate Jordan mappings. A linear mapping f from a normed algebra A into a normed algebra B is called an ε-approximate Jordan mapping if for all a in A

$$\left\|f(a^2) - f(a)^2\right\| \leq \varepsilon \|a\|^2 .$$

If B is the complex field, then f is called an ε-appoximate Jordan functional. For ε-appoximate mappings the reader is referred to [3],[4],[5],[6],[9],[10],[11].

A linear mapping f is a strong ε-approximate Jordan mapping if $||f(a^2) - f(a)^2|| < \varepsilon$. Also a continuous linear mapping f between normed algebras is an ε-near Jordan mapping if $\|f - J\| \leq \varepsilon$ for some continuous Jordan mapping J. In this paper, we prove that every strong ε-approximate one-to-one Jordan functional on an

2000 *Mathematics Subject Classification.* Primary 39B82, Secondary 46H40, 46J10.
Key words and phrases. Banach algebra, Automatic continuity, Jordan mapping, Superstability.

algebra is a Jordan functional and every ε-approximate one-to-one Jordan functional on a Banach algebra is continuous.

2. Main Results

Theorem 1. *If f is a strong ε-approximate one-to-one Jordan functional on an algebra A, then f is a Jordan functional. In particular if A is a Banach algebra, then f is continuous.*

Proof. Since, for every $x, y \in A$, $|f\left((x+y)^2\right) - f(x+y)^2| \leq \varepsilon$, we have $|f(xy + yx) - 2f(x)f(y)| \leq 3\varepsilon$. If x and y are commute, $|f(xy) - f(x)f(y)| \leq \frac{3\varepsilon}{2}$. Now we use the method of the proof in [1]. Let $c(\varepsilon) = \frac{1+\sqrt{1+4\varepsilon}}{2}$. Note that $c(\varepsilon)^2 - c(\varepsilon) = \varepsilon$ and $c(\varepsilon) > 1$. Let $a \in A$. If $a \neq 0$ we may assume that $|f(a)| > c(\varepsilon)$ because $|f(ta)| > c(\varepsilon)$ for some $t \in R$ and $f((ta)^2) = f(ta)^2$ implies $f(a^2) = f(a)^2$. Say $|f(a)| = c(\varepsilon) + p$ for some $p > 0$. Then

$$\begin{aligned} |f(a^2)| &= |f(a)^2 - (f(a)^2 - f(a^2))| \geq |f(a)^2| - |(f(a)^2 - f(a^2))| \\ &\geq (c(\varepsilon) + p)^2 - \varepsilon > c(\varepsilon) + 2p. \end{aligned}$$

By induction, $|f(a^{2^n})| > c(\varepsilon) + (n+1)p$ for all $n = 1, 2, 3, \cdots$. For every $x, y, z \in A$ which they are commute, $|f(xyz) - f(xy)f(z)| \leq \frac{3\varepsilon}{2}$ and $|f(xyz) - f(x)f(yz)| \leq \frac{3\varepsilon}{2}$. So $|f(xy)f(z) - f(x)f(yz)| \leq 3\varepsilon$. Hence

$$|f(xy)f(z) - f(x)f(y)f(z)|$$

$$\leq |f(xy)f(z) - f(x)f(yz)| + |f(x)f(yz) - f(x)f(y)f(z)| \leq 3\varepsilon + |f(x)|\frac{3\varepsilon}{2}.$$

By letting $x = a, y = a$ and $z = a^{2^n}$, we have

$$|f(a^2) - f(a)^2| \leq \frac{3\varepsilon + |f(a)|\frac{3\varepsilon}{2}}{|f(a^{2^n})|}.$$

Letting $n \longrightarrow +\infty$ shows that $f(a^2) = f(a)^2$.

Theorem 2. *Let f be an ε-approximate Jordan functional on a normed algebra A with the multiplicative norm. Then for each $a \in A$, either $|f(a)| \leq \frac{1+\sqrt{1+4\varepsilon}}{2}\|a\|$ or $f(a^2) = f(a)^2$.*

Proof. Let $a \in A$ and $c = \frac{a}{\|a\|}$. If $|f(a)| > \frac{1+\sqrt{1+4\varepsilon}}{2}\|a\|$ then $|f(c^{2^n})| > c(\varepsilon) + (n+1)p$ for all $n = 1, 2, 3$ and for some p, where $c(\varepsilon) = \frac{1+\sqrt{1+4\varepsilon}}{2}$, by the proof of Theorem 1.

For any natural number m, n,

$$|f(c^n c^m) - f(c^n)f(c^m)|$$

$$\le |f((c^n + c^m)^2) - f(c^n + c^m)^2| + |f((c^n)^2) - f(c^n)^2| + |f((c^m)^2) - f(c^m)^2|$$

$$\le \frac{\varepsilon}{2}\left(\|c^n + c^m\|^2 + \|c^n\|^2 + \|c^m\|^2\right) = 3\varepsilon.$$

Then we have

$$|f(c^2) - f(c)^2| \le \frac{1}{|f(c^{2^n})|}\big(|f((c^2)f(c^{2^n}) - f(c^2 + c^{2^n})|$$

$$+|f(c^2 \cdot c^{2^n}) - f(c^2)f(c^{2^n})| + |f(c)||f(c \cdot c^{2^n}) - f(c)f(c^{2^n})|\big)$$

$$\le \frac{6\varepsilon + 3|f(c)|\varepsilon}{|f(c^{2^n})|} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

This shows that $f(a^2) = f(a)^2$.

Corollary 3. *Let S be a compact Hausdorff space and $C(S)$ the set of all continuous complex valued functions. If f is an ε-approximate Jordan mapping from a Banach algebra A with the multiplicative norm into $C(S)$, then for each $a \in A$, either $\|f(a)\| \le \frac{1+\sqrt{1+4\varepsilon}}{2}\|a\|$ or $f(a^2) = f(a)^2$.*

Proof. For every $x \in S$, we can define a linear functional $f_x : A \longrightarrow C$ by $f_x(a) = f(a)(x)$ for all $a \in A$. Then for every $a \in A$,

$$|f_x(a^2) - f_x(a)^2| \le \left\|f(a^2) - f(a)^2\right\| \le \varepsilon \|a\|^2 .$$

By Theorem 2, either $\|f_x(a)\| \le \frac{1+\sqrt{1+4\varepsilon}}{2}$ or $f_x(a^2) = f_x(a)^2$ for any $a \in A$. Then we complte the proof.

In Theorem 2 and Corollary 3 we used the assumption that an algebra A has the multiplicative norm. It is not known that whether they hold or not without such condition. With another condition we obtain the following theorem.

Theorem 4. *Let f be an ε-approximate Jordan functional on a Banach algebra A such that $f(a) = 0$ implies $f(a^2) = 0$ for each $a \in A$. Then f is continuous and $\|f\| \le \frac{1+\sqrt{1+4\varepsilon}}{2}$.*

Proof. If A does not posses a unit, then we can extened f to $A \oplus (\lambda 1)$ by putting $f(a \bigoplus \lambda 1) = f(a) + \lambda$, and the extended f is still an ε -approximate Jordan functional. Thus without loss of generality we may assume that A has a unit. Suppose that f is discontinuous. Then the kernel $Ker(f)$ of f is a dense subset of A. Since the unit element 1 is the closure of $Ker(f)$, we can choose $c \in Ker(f)$ such that $\|c-1\| \leq \frac{1}{3}$. Then c is invertible, and $c^{-1} = 1 + \sum_{n=1}^{\infty}(1-c)^n$. And so $\left\|c^{-1}\right\| \leq \frac{1}{1-\|c-1\|} \leq \frac{3}{2}$. Let $b = \frac{c}{\|c\|} \in Ker(f)$. Then $b^{-1} = \|c\| c^{-1}$and $\left\|b^{-1}\right\| \leq 2$. Put $|f(b^{-1})| = \alpha$. Note that for every $x, y \in A$

$$|f(xy+yz) - 2f(x)f(y)| \leq |f((x+y)^2) - (f(x+y))^2|$$

$$+|f(x^2) - f(x)^2| + |f(y^2) - f(y)^2| \leq 2\varepsilon(\|x\|^2 + \|y\|^2 + \|x\|\,\|y\|).$$

If b^{-1} is not in $Ker(f)$, then for every a in A with $\|a\| = 1$,

$$\begin{aligned} |f(a)| &= \frac{1}{2\alpha}|2f(a)f(b^{-1})| \\ &\leq \frac{1}{2\alpha}(|2f(a)f(b^{-1}) - f(ab^{-1} + b^{-1}a)| \\ &+ |f(bb^{-1}ab^{-1} + b^{-1}ab^{-1}b) - 2f(b^{-1}ab^{-1})f(b)| \leq \frac{28\varepsilon}{\alpha}. \end{aligned}$$

Thus f is bounded and it is a contradiction. Therefore b^{-1} is in $Ker(f)$. By assumption, b^{-2} is in $Ker(f)$. Then for every a in A with $\|a\| = 1$,

$$|f(a)| = \frac{1}{2}(|f(a + b^{-1}ab)| + |f(a + bab^{-1})| + |f(b^{-1}ab + bab^{-1})|)$$

$$= \frac{1}{2}(|f(a + b^{-1}ab) - 2f(b^{-1}a)f(b)| + |f(a + bab^{-1}) - 2f(ab^{-1})f(b)|$$

$$+|f(b^{-1}ab + bab^{-1}) - 2f(bab)f(b^{-2})|) \leq 35\varepsilon.$$

Thus f is continuous. Since $|f(a^2) - f(a)^2| < \varepsilon$ for every $a \in A$ with $\|a\| = 1$, $|f(a^2)| - \varepsilon \leq |f(a^2)| \leq \|f\|$ and consequently $\|f\| \geq \|f\|^2 - \varepsilon$. This proves $\|f\| \leq \frac{1+\sqrt{1+4\varepsilon}}{2}$.

Corollary 5. *Every ε-approximate one-to-one Jordan functional on a Banach algebra is continuous and its norm is less than or equal to $\frac{1+\sqrt{1+4\varepsilon}}{2}$.*

Let f be an ε-near Jordan mapping from a Banach algebra A into a Banach algebra B. Then there exists a Jordan mapping J such that $\|f - J\| \leq \epsilon$. For every a in A,

$$\begin{aligned} \left\|f(a^2) - f(a)^2\right\| &\leq \left\|f(a^2) - J(a^2)\right\| + \left\|f(a)^2 - J(a)^2\right\| \\ &\leq \varepsilon\|a\|^2 + \|f(a) - J(a)\|\,\|f(a)\| + \|J(a)\|\,\|f(a) - J(a)\| \\ &\leq (\varepsilon + \varepsilon\|f\| + \varepsilon\|J\|)\,\|a\|^2. \end{aligned}$$

Therefore f is a $\varepsilon(1 + \|f\| + \|J\|)$-approximate Jordan mapping. We are concerned with it's converse. By the method of the proof in [8] we obtain the following theorem.

Theorem 6. *For every $\varepsilon > 0$ and $K > 0$, there exists a positive integer m such that every $\frac{\varepsilon}{m}$-approximate Jordan mapping with norm less than or equal to K on a finite demensional Banach algebra A is an ε- near Jordan mapping.*

Proof. Let $J(A)$ be the set of all bounded Jordan mapping on a finite dimensional Banach algebra $A, BL(A)$ the set of all bounded linear mappings on A, and let for each f in $BL(A)$

$$N(f) = \inf\{\|f - J\| : J \in J(A)\},$$

$$M = \{f \in BL(A) : N(f) \geq \varepsilon \text{ and } \|f\| \leq k\}$$

and

$$G_n = \left\{ f \in BL(A) : \sup_{\|a\| \leq 1} \left\| f(a^2) - f(a)^2 \right\| \geq \frac{\varepsilon}{n} \right\}.$$

Since M is a closed and bounded subset of a finite dimensional space $BL(A)$, M is compact. Since G_n is open for each n and

$$M \subset BL(A) \setminus J(A) \subset \bigcup_{n=1}^{\infty} G_n,$$

there is m such that $M \subset G_m$. If $f \in BL(A) \setminus G_m$, then $f \in BL(A) \setminus M$. Therefore if f is an $\frac{\varepsilon}{m}$- approximate Jordan mapping then f is an ε- near Jordan mapping.

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Received: 23.05.2000