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CERTAIN SUBCLASSES OF MEROMORPHIC UNIVALENT FUNCTIONS WITH MISSING AND TWO FIXED POINTS

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Abstract. The systematic study of some novel subclasses $\Omega^*_{pi}(\alpha, \beta, \mu, z_0)$, $(i = 0, 1)$ consisting functions of the type

$$f(z) = a_0 z^{-1} + \sum_{n=0}^{\infty} a_{p+n} z^{p+n},\ a_0 > 0, a_{p+n} \geq 0, p \in N$$

which are meromorphic and univalent in $U^* = \{z : 0 < |z| < 1\}$ is presented here. The various results for example coefficient estimates, radius of convexity, distortion theorem are obtained for $f(z)$ to be in the above mentioned classes.

1. **Introduction and Definitions**

Let Ω denote the class of functions of the form

$$f(z) = a_0 z^{-1} + \sum_{n=1}^{\infty} a_n z^n,\ a_0 > 0 \tag{1.1}$$

which are analytic in the punctured disk $U^* = \{z : 0 < |z| < 1\}$. Further, Ω^* is the class of all functions in Ω which are univalent in U^*. We denote by Ω^*_p, a subclass of Ω^* consisting functions of the form

$$f(z) = a_0 z^{-1} + \sum_{n=0}^{\infty} a_{p+n} z^{p+n},\ a_0 > 0, a_{p+n} > 0, p \in N, \tag{1.2}$$

$$N = \{1, 2, 3, \cdots\}.$$

Definition. A function $f(z)$ belonging to the class Ω^*_p is in the class $\Omega^*_p(\alpha, \beta, \mu)$ if it satisfies the condition

$$\left| \frac{z^2 f'(z) + a_0}{\mu z^2 f'(z) - a_0 + (1+\mu)\alpha a_0} \right| < \beta, \tag{1.3}$$

for some $0 \leq \alpha < 1, 0 < \beta \leq 1$ and $0 \leq \mu \leq 1$.

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For a given real number $z_0 (0 < z_0 < 1)$. Let $\Omega_{pi}(i = 0, 1)$ be a subclass of Ω_p^* satisfying the condition $z_0 f(z_0) = 1$ and $-z_0^2 f'(z_0) = 1$ respectively.

Let

$$\Omega_{pi}^*(\alpha, \beta, \mu, z_0) = \Omega_p^*(\alpha, \beta, \mu) \cap \Omega_{pi} \ (i = 0, 1). \tag{1.4}$$

In our systematic investigation of the various properties and characteristics of the class $\Omega_{pi}^*(\alpha, \beta, \mu)$, we shall require use of number of other classes of functions associated with Ω_p^*. First of all, a function $f \in \Omega_p^*$ is said to be meromorphic starlike of order α in U^* if it satisfies the inequality

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > -\alpha, \ z \in U^*, 0 \le \alpha < 1. \tag{1.5}$$

On the other hand, a function $f \in \Omega_p^*$ is said to convex of order α in U, if it satisfies the inequality

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > -\alpha, \ z \in U^*, 0 \le \alpha < 1. \tag{1.6}$$

For other subclasses of meromorphic univalent function, one may refer to the recent work of Aouf [1], Aouf and Darwish [2], Cho *et al* [3], Joshi *et al* [4], Srivastava and Owa [5]. In the present paper we obtain coefficient estimates, distortion theorems, closure theorems and radius of convexity of order $\delta (0 \le \delta < 1)$ for the classes $\Omega_{pi}^*(\alpha, \beta, \mu, z_0)(i = 0, 1)$. Further, we look for necessary and sufficient condition that a subset B of the real interval $[0, 1]$ should satisfy the property $\cup_{z_r \in B} \Omega_{p0}^*(\alpha, \beta, \mu, z_r)$ and $\cup_{z_r \in B} \Omega_{p1}(\alpha, \beta, \mu, z_r)$ each forms a convex family. The techniques used are similar to Uralegaddi and Ganigi [6].

2. **Main Results**

Coefficient Estimates

Theorem 1. Let the function $f(z)$ be defined by (1.2) is in the class $\Omega_p^*(\alpha, \beta, \mu)$ if and only if

$$\sum_{n=0}^{\infty} (p + n)(1 + \mu\beta) a_{p+n} \le \beta a_0 (1 - \alpha)(1 + \mu). \tag{2.1}$$

The result is sharp and is given by

$$f(z) = \frac{a_0}{z} + \frac{\beta(1 - \alpha)(1 + \mu) a_0 z^{p+n}}{(p + n)(1 + \mu\beta)} \ , n \ge 1. \tag{2.2}$$

Proof. The proof of Theorem 1 is straightforward, hence omitted.

Theorem 2. Let the function $f(z)$ be defined by (1.2). Then $f(z) \in \Omega^*_{p0}(\alpha,\beta,\mu,z_0)$ if and only if

$$\sum_{n=0}^{\infty}\left[\frac{(p+n)(1+\mu\beta)}{\beta(1-\alpha)(1+\mu)}+z_0^{p+n+1}\right]\ a_{p+n}\leq 1. \tag{2.3}$$

Proof. Since $f(z) \in \Omega^*_{p0}(\alpha,\beta,\mu,z_0)$, we have

$$z_0 f(z_0) = a_0 + \sum_{n=0}^{\infty} a_{p+n} z_0^{p+n+1},\ a_0 \geq 0,\ a_{p+n} \geq 0,$$

which gives

$$a_0 = 1 - \sum_{n=0}^{\infty} a_{p+n} z_0^{p+n+1} \tag{2.4}$$

substituting this value of a_0 (given by (2.4)) in Theorem 1, we get the desire assertion.

Theorem 3. Let the function $f(z)$ be defined (1.2). Then $f(z) \in \Omega^*_{p1}(\alpha,\beta,\mu,z_0)$ if and only if

$$\sum_{n=0}^{\infty}(p+n)\left[\frac{(1+\mu\beta)}{\beta(1-\alpha)(1+\mu)}-z_0^{p+n+1}\right]\ a_{p+n}\leq 1. \tag{2.5}$$

Proof. Since $-z_0^2 f'(z_0) = 1$, we have

$$a_0 = 1 + \sum_{n=0}^{\infty}(p+n) a_{p+n} z_0^{p+n+1} \tag{2.6}$$

Eliminating a_0 from (2.1) and (2.6) we get the required result.

An immediate consequence of Theorem 2 and Theorem 3 may be stated as the following.

Corollary 1. Let, $f(z)$ given by (1.2) be in the class $\Omega^*_{p0}(\alpha,\beta,\mu,z_0)$ then

$$a_{p+n} \leq \frac{\beta(1+\mu)(1-\alpha)}{(p+n)(1+\mu\beta)+\beta(1+\mu)(1-\alpha)z_0^{p+n+1}}. \tag{2.7}$$

The equality in the (2.7) is attained for the function $f(z)$ given by

$$f(z) = \frac{(p+n)(1+\mu\beta)+\beta(1+\mu)(1-\alpha)z^{p+n+1}}{z[(p+n)(1+\mu\beta)+\beta(1+\mu)(1-\alpha)z_0^{p+n+1}]}, \tag{2.8}$$
$$p \in N, n \geq 0.$$

Corollary 2. Let the function $f(z)$ given by (1.2) in the class $\Omega^*_{p1}(\alpha,\beta,\mu,z_0)$ then

$$a_{p+n} \leq \frac{\beta(1+\mu)(1-\alpha)}{(p+n)[(1+\mu\beta)-\beta(1+\mu)(1-\alpha)z_0^{p+n+1}]}. \tag{2.9}$$

The equality holds for the function $f(z)$ given by

$$f(z) = \frac{(p+n)(1+\mu\beta)+\beta(1+\mu)(1-\alpha)z^{p+n+1}}{z(p+n)[(1+\mu\beta)-\beta(1+\mu)(1-\alpha)z_0^{p+n+1}]}. \tag{2.10}$$

3. **Distortion Theorem**

In this section, we prove distortion theorem associated with the classes introduced in section 1, we first state the following theorem.

Theorem 4. Let $f(z) \in \Omega^*_{p0}(\alpha, \beta, \mu, z_0)$ then,

$$|f(z)| \geq \frac{p(1+\mu\beta) - \beta(1+\mu)(1-\alpha)r^{p+1}}{r[p(1+\mu\beta) + \beta(1+\mu)(1-\alpha)z_0^{p+1}]}, \tag{3.1}$$

for $0 < |z| = r < 1$. The result is sharp.

Proof. Since $f \in \Omega^*_{p0}(\alpha, \beta, \mu, z_0)$, by applying assertion (2.3) of Theorem 2, we obtain

$$\sum_{n=0}^{\infty} a_{p+n} \leq \frac{\beta(1+\mu)(1-\alpha)}{p(1+\mu\beta) + \beta(1+\mu)(1-\alpha)z_0^{p+1}}. \tag{3.2}$$

Further from (2.4), we have

$$\begin{aligned} a_0 &= 1 - \sum_{n=0}^{\infty} a_{p+n} z_0^{p+n+1} \\ &\geq \frac{(1+\mu\beta)p}{p(1+\mu\beta) + \beta(1+\mu)(1-\alpha)z_0^{p+1}}. \end{aligned} \tag{3.3}$$

Hence we have

$$\begin{aligned} |f(z)| &\geq a_0 r^{-1} - r^p \sum_{n=0}^{\infty} a_{p+n} \\ &\geq \frac{p(1+\mu\beta) - \beta(1+\mu)(1-\alpha)r^{p+1}}{r[p(1+\mu\beta) + \beta(1+\mu)(1-\alpha)z_0^{p+1}]}, \end{aligned} \tag{3.4}$$

by using (3.2) and (3.3). Further, the result is sharp for the function $f(z)$ given by

$$f(z) = \frac{p(1+\mu\beta) - \beta(1+\mu)(1-\alpha)z^{p+1}}{z[p(1+\mu\beta) + \beta(1+\mu)(1-\alpha)z_0^{p+1}]}. \tag{3.5}$$

Theorem 5. If $f(z) \in \Omega^*_{p1}(\alpha, \beta, \mu, z_0)$ then

$$|f(z)| \leq \frac{p(1+\mu\beta) + \beta(1+\mu)(1-\alpha)r^{p+1}}{r[p(1+\mu\beta) - \beta(1+\mu)(1-\alpha)z_0^{p+1}]} \tag{3.6}$$

for $0 < |z| = r < 1$. The result is sharp.

Proof. It follows from assertion (2.5) of Theorem 3, that

$$\sum_{n=0}^{\infty} a_{p+n} \leq \frac{\beta(1+\mu)(1-\alpha)}{p[(1+\mu\beta) - \beta(1+\mu)(1-\alpha)z_0^{p+1}]} \tag{3.7}$$

and

$$\sum_{n=0}^{\infty} (p+n)a_{p+n} \leq \frac{\beta(1+\mu)(1-\alpha)}{[(1+\mu\beta) - \beta(1+\mu)(1-\alpha)z_0^{p+1}]}. \tag{3.8}$$

From (2.6) we have

$$a_0 = 1 + \sum_{n=0}^{\infty}(p+n)a_{p+n}z_0^{p+n+1} \tag{3.9}$$
$$\leq \frac{(1+\mu\beta)}{[(1+\mu\beta) - \beta(1+\mu)(1-\alpha)z_0^{p+1}]}.$$

Hence we have

$$|f(z)| \leq a_0 r^{-1} + r^{p+1}\sum_{n=0}^{\infty} a_{p+n}$$
$$\leq \frac{p(1+\mu\beta) + \beta(1+\mu)(1-\alpha)r^{p+1}}{rp[(1+\mu\beta) - \beta(1+\mu)(1-\alpha)z_0^{p+1}]}, \tag{3.10}$$

by using (3.7) and (3.9). Further the result is sharp for the function given by

$$f(z) = \frac{p(1+\mu\beta) + \beta(1+\mu)(1-\alpha)z^{p+1}}{zp[(1+\mu\beta) - \beta(1+\mu)(1-\alpha)z_0^{p+1}]}. \tag{3.11}$$

4. **Closure Theorems**

Let the functions $f_j(z)$ be defined, for $j = 1, 2, \cdots, m$ by

$$f_j(z) = \frac{a_{0,j}}{z} + \sum_{n=0}^{\infty} a_{p+n,j}z^{p+n} \ (a_{0,j} > 0, a_{p+n,j} \geq 0) \ z \in U^*. \tag{4.1}$$

Theorem 6. Let $f_j(z)$ defined by (4.1) be in the class $\Omega^*_{p0}(\alpha, \beta, \mu, z_0)$. Then the function $h(z)$ defined by

$$h(z) = \sum_{j=0}^{m} d_j f_j(z), \ (d_j \geq 0) \tag{4.2}$$

is also in the same class $\Omega^*_{p0}(\alpha, \beta, \mu, z_0)$, where

$$\sum_{j=0}^{m} d_j = 1. \tag{4.3}$$

Proof. According to the definition (4.2) we have

$$h(z) = \frac{b_0}{z} + \sum_{n=0}^{\infty} b_{p+n}z^{p+n}, \tag{4.4}$$

where

$$b_0 = \sum_{j=0}^{m} d_j a_{0,j} \text{ and } b_{p+n} = \sum_{j=0}^{m} d_j a_{p+n,j}, (n = 0, 1, 2, \cdots, m).$$

Since $f_j(z) \in \Omega^*_{p0}(\alpha, \beta, \mu, z_0)$ $(j = 0, 1, 2, \cdots, m)$, using Theorem 2 we have

$$\sum_{n=0}^{\infty}\{(p+n)(1+\mu\beta) + \beta(1-\alpha)(1+\mu)z_0^{p+n+1}\} \leq \beta(1-\alpha)(1+\mu)$$

for every $j = 0, 1, \cdots, m$. Therefore we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\{(p+n)(1+\mu\beta)+\beta(1-\alpha)(1+\mu)z_0^{p+n+1}\}(\sum_{j=0}^{m} d_j a_{p+n,j})\\
=&\sum_{j=0}^{m} d_j\{\sum_{n=0}^{\infty}[(p+n)(1+\mu\beta)+\beta(1-\alpha)(1+\mu)z_0^{p+n+1}]a_{p+n,j}\}\\
\leq&(\sum_{j=0}^{m} d_j)\beta(1-\alpha)(1+\mu)\\
=&\beta(1-\alpha)(1+\mu)
\end{aligned}$$

which shows that $h(z) \in \Omega^*_{p0}(\alpha, \beta, \mu, z_0)$.

Theorem 7. Let the functions $f_j(z)(j = 0, 1, \cdots, m)$ defined by (4.1) be in the class $\Omega^*_{p1}(\alpha, \beta, \mu, z_0)$ for every $j = 0, 1, \cdots, m$. Then the function $h(z)$ defined by (4.2) is also in the same class $\Omega^*_{p1}(\alpha, \beta, \mu, z_0)$, under the assumption (4.3).

Proof. The proof of Theorem 7, can be given on using the same techniques as in the proof of Theorem 6, using Theorem 3.

Theorem 8. The class $\Omega^*_{p0}(\alpha, \beta, \mu, z_0)$ is closed under convex linear combination.

Proof. Let $f_j(z)(j = 0, 1, \cdots, m)$ defined by (4.1) be in the class $\Omega^*_{p0}(\alpha, \beta, \mu, z_0)$, it is sufficient to show that the function $H(z)$ defined by

$$H(z) = \lambda f_1(z) + (1-\lambda) f_2(z),\ 0 \leq \lambda \leq 1, \tag{4.5}$$

is also in the class $\Omega^*_{p0}(\alpha, \beta, \mu, z_0)$. Since

$$H(z) = \frac{\lambda a_{0,1} + (1-\lambda)a_{0,2}}{z} + \sum_{n=0}^{\infty}\{\lambda a_{p+n,1} + (1-\lambda)a_{p+n,2}\}z^{p+n}$$

with the aid of Theorem 2, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\{(p+n)(1+\mu\beta)+\beta(1-\alpha)(1+\mu)z_0^{p+n+1}\}[\lambda a_{p+n,1}+(1-\lambda)a_{p+n,2}]\\
\leq \beta(1-\alpha)(1+\mu)
\end{aligned} \tag{4.6}$$

which implies that $H(z) \in \Omega^*_{p0}(\alpha, \beta, \mu, z_0)$. In a similar manner, by using Theorem 3, we can prove the following Theorem.

Theorem 9. The class $\Omega^*_{p1}(\alpha, \beta, \mu, z_0)$ is closed under convex linear combination.

Theorem 10. Let

$$f_0(z) = 1/z \tag{4.7}$$

and

$$f_{p+n}(z)=\frac{(p+n)(1+\mu\beta)+\beta(1+\mu)(1-\alpha)z^{p+n+1}}{z[(p+n)(1+\mu\beta)+\beta(1+\mu)(1-\alpha)z_0^{p+n+1}]}\ ,n\geq 0 \qquad (4.8)$$

then $f(z)$ is in the class $\Omega^*_{p0}(\alpha,\beta,\mu,z_0)$, if and only if it can be expressed in the form:

$$f(z)=\sum_{n=0}^{\infty}\lambda_n f_n(z),\ \text{ where } \lambda_n\geq 0, \qquad (4.9)$$

$$\lambda_i=0(i=1,2,\cdots,p-1,p\geq 2)\ \text{ and }\ \sum_{n=0}^{\infty}\lambda_n=1. \qquad (4.10)$$

Proof. Assume that

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}\lambda_n f_n(z)\\ &= \lambda_0/z+\sum_{n=0}^{\infty}\frac{[(p+n)(1+\mu\beta)+\beta(1+\mu)(1-\alpha)z^{p+n+1}]\lambda_{p+n}}{z[(p+n)(1+\mu\beta)+\beta(1+\mu)(1-\alpha)]z_0^{p+n+1}}\\ &= \frac{1}{z}\left[\lambda_0+\sum_{n=0}^{\infty}\frac{(p+n)(1+\mu\beta)\lambda_{p+n}}{[(p+n)(1+\mu\beta)+\beta(1+\mu)(1-\alpha)z_0^{p+n+1}]}\right]\\ &\quad+\sum_{n=0}^{\infty}\frac{\beta(1+\mu)(1-\alpha)\lambda_{n+p}z^{p+n}}{(p+n)(1+\mu\beta)+\beta(1+\mu)(1-\alpha)z_0^{p+n+1}} \end{aligned}$$

Then it follows from theorem 2, that

$$\sum_{n=0}^{\infty}\frac{(p+n)(1+\mu\beta)+\beta(1+\mu)(1-\alpha)z_0^{p+n+1}}{\beta(1+\mu)(1-\alpha)}\ \frac{\beta(1+\mu)(1-\alpha)\lambda_{p+n}}{(p+n)(1+\mu\beta)+\beta(1+\mu)(1-\alpha)z_0^{p+n+1}}$$

$$=\sum_{n=0}^{\infty}\lambda_{p+n}=1-\lambda_0\leq 1.$$

Also by definition we have $z_0 f_{p+n}(z_0)=1$. Therefore

$$z_0 f(z_0)=\sum_{n=0}^{\infty}\lambda_{p+n}z_0 f_{p+n}(z_0)=\sum_{n=0}^{\infty}\lambda_{p+n}=1.$$

This implies $f\in\Omega_{p0}$, so by theorem 2, $f(z)\in\Omega^*_{p0}(\alpha,\beta,\mu,z_0)$.

Conversely, assume that the function $f(z)$ given by (1.2) belongs to the class $\Omega^*_{p0}(\alpha,\beta,\mu,z_0)$. Then

$$a_{p+n}\leq\frac{\beta(1+\mu)(1-\alpha)}{(p+n)(1+\mu\beta)+\beta(1+\mu)(1-\alpha)z_0^{p+n+1}},n\geq 0. \qquad (4.11)$$

Setting

$$\lambda_{p+n}=\frac{[(p+n)(1+\mu\beta)+\beta(1+\mu)(1-\alpha)z_0^{p+n+1}]}{\beta(1+\mu)(1-\alpha)}\ a_{p+n},\ n\geq 0$$

and

$$\lambda_0 = 1 - \sum_{n=0}^{\infty} \lambda_{p+n}.$$

Hence, it is observed that $f(z)$ can be expressed in the form (4.9). This completes the proof of Theorem 10.

In a similar manner, we can prove the following Theorem.

Theorem 11. Define

$$f_0(z) = \frac{1}{z} \tag{4.12}$$

and

$$f_{p+n}(z) = \frac{(p+n)(1+\mu\beta) + \beta(1+\mu)(1-\alpha)z^{p+n+1}}{z(p+n)[(1+\mu\beta) - \beta(1+\mu)(1-\alpha)z_0^{p+n+1}]},\ n \geq 0 \tag{4.13}$$

then $f(z)$ is in the class $\Omega^*_{p1}(\alpha, \beta, \mu, z_0)$ if and only if it can be expressed in the form (4.9) where $\lambda_n \geq 0$ and (4.10).

5. **Radius of Convexity**

In this section we determine the radius of convexity of order $\delta(0 \leq \delta < 1)$ for the class $\Omega^*_{pi}(\alpha, \beta, \mu, z_0)(i = 0, 1)$.

Theorem 12. Let the function defined by (1.2) be in the class $\Omega^*_{p0}(\alpha, \beta, \mu, z_0)$ or $\Omega^*_{p1}(\alpha, \beta, \mu, z_0)$, then $f(z)$ is convex of order $\delta(0 \leq \delta < 1)$ in $0 < |z| < R^*(\alpha, \beta, \mu, \delta)$ where

$$R^*(\alpha, \beta, \mu, \delta) = \inf_n \left[\frac{(1-\delta)(1+\mu\beta)}{(1-\alpha)\beta(1+\mu)(p+n+2-\delta)} \right]^{1/(p+n+1)}, n \geq 0. \tag{5.1}$$

The result (5.1) is sharp.

Proof. It is sufficient to show that

$$\left| 1 + \frac{zf''(z)}{f'(z)} \right| \leq (1-\delta),\ 0 \leq \delta < 1,$$

for $0 < |z| < R^*(\alpha, \beta, \mu, \delta)$.

We have

$$\left| \frac{f'(z) + [zf'(z)]'}{f'(z)} \right| \leq \sum_{n=0}^{\infty} \frac{(p+n)(p+n+1)a_{p+n}|z|^{p+n+1}}{a_0 - \sum_{n=0}^{\infty}(p+n)a_{p+n}|z|^{p+n+1}}.$$

Thus

$$\left| \frac{f'(z) + [zf'(z)]'}{f'(z)} \right| \leq (1-\delta)$$

if

$$\sum_{n=0}^{\infty}(p+n)(p+n+2-\delta)a_{p+n}|z|^{p+n+1} \leq (1-\delta)a_0 \tag{5.2}$$

when $f(z) \in \Omega^*_{p0}(\alpha, \beta, \mu, z_0)$, using (2.4) we find that inequality (5.2) is equivalent to

$$\sum_{n=0}^{\infty}\{(p+n)(p+n+2-\delta)|z|^{p+n+1} + (1-\delta)z_0^{p+n+1}\}a_{p+n} \leq (1-\delta). \tag{5.3}$$

But Theorem 2 ensures

$$\sum_{n=0}^{\infty}(1-\delta)\left[\frac{(p+n)(1+\mu\beta)}{\beta(1-\alpha)(1+\mu)} + z_0^{p+n+1}\right]a_{p+n} \leq (1-\delta). \tag{5.4}$$

Hence (5.3) holds if

$$\begin{aligned}&\{(p+n)(n+p+2-\delta)|z|^{p+n+1} + (1-\delta)z_0^{p+n+1}\}a_{p+n}\\ &\leq \left\{(1-\delta)\left[\frac{(p+n)(1+\mu\beta)}{\beta(1-\alpha)(1+\mu)} + z_0^{p+n+1}\right]\right\}a_{p+n}, n \geq 0,\end{aligned}$$

or if

$$|z| \leq \left[\frac{(1-\delta)(1+\mu\beta)}{(1-\alpha)\beta(1+\mu)(p+n+2-\delta)}\right]^{1/(p+n+1)}, \ n \geq 0.$$

Thus $f(z)$ is convex of order $\delta(0 \leq \delta < 1)$ in $0 < |z| < R^*(\alpha, \beta, \mu, \delta)$.

In other case when $f(z) \in \Omega^*_{p1}(\alpha, \beta, \mu, z_0)$ using (2.6) we find that the inequality (5.2) is equivalent to

$$\sum_{n=0}^{\infty}(p+n)[(p+n+2-\delta)|z|^{p+n+1} - (1-\delta)z_0^{p+n+1}]a_{p+n} \leq (1-\delta). \tag{5.5}$$

Therefore, in view of Theorem 3, the inequality (5.5) holds if

$$\begin{aligned}&(p+n)[(p+n+2-\delta)|z|^{p+n+1} - (1-\delta)z_0^{p+n+1}]a_{p+n}\\ &\leq (1-\delta)(p+n)\left[\frac{(1+\mu\beta)}{(1-\alpha)\beta(1+\mu)} - z_0^{p+n+1}\right]a_{p+n}\end{aligned}$$

or if

$$|z| \leq \left[\frac{(1-\delta)(1+\mu\beta)}{(1-\alpha)\beta(1+\mu)(p+n+2-\delta)}\right]^{1/(p+n+1)}, \ n \geq 0.$$

This completes the proof of theorem 12.

Sharpness for the class $\Omega^*_{p0}(\alpha, \beta, \mu, z_0)$ follows by taking the functions $f(z)$ given by (2.8), whereas for the class $\Omega^*_{p1}(\alpha, \beta, \mu, z_0)$, sharpness follows if we take the function given by (2.10).

Remark. The conclusion of Theorem 12 is independent of z_0.

6. **Convex Family**

Let B be a nonempty subset of a real interval $[0,1]$. We define a family $\Omega^*_{p0}(\alpha,\beta,\mu,B)$ by

$$\Omega^*_{p0}(\alpha,\beta,\mu,B)=\cup_{z_r\in B}\Omega^*_{p0}(\alpha,\beta,\mu,z_r).$$

If B has only one element, then $\Omega^*_{p0}(\alpha,\beta,\mu,B)$ is known to be a convex family by Theorems 6 and 8. It is interesting to investigate this class for other subset B.

We shall make use of the following

Lemma 1. If $f(z)\in\Omega^*_{p0}(\alpha,\beta,\mu,z_0)\cap\Omega^*_{p0}(\alpha,\beta,\mu,z_1)$ where z_0 and z_1 are distinct positive numbers then $f(z)=1/z$.

Proof. If $f(z)\in\Omega^*_{p0}(\alpha,\beta,\mu,z_0)\cap\Omega^*_{p0}(\alpha,\beta,\mu,z_1)$ and let

$$f(z)=a_0z^{-1}+\sum_{n=0}^{\infty}a_{p+n}z^{p+n},\ a_0>0,a_{p+n}>0,p\in N,$$

then

$$a_0=1-\sum_{n=0}^{\infty}a_{p+n}z_0^{p+n+1}=1-\sum_{n=0}^{\infty}a_{p+n}z_1^{p+n+1}$$

since $a_{p+n}\geq 0, z_0>0$ and $z_1>0$, this implies $a_{p+n}\equiv 0$ for each $n\geq 0$ and $f(z)=1/z$. Hence the proof of lemma is complete.

Theorem 13. If B is contained in the interval $[0,1]$, then $\Omega^*_{p0}(\alpha,\beta,\mu,B)$ is a convex family if and only if B is connected.

Proof. Suppose B is connected and $z_0,z_1\in B$ with $z_0\leq z_1$. To prove $\Omega^*_{p0}(\alpha,\beta,\mu,B)$ is a convex family it suffices to show, for

$$\begin{aligned}f(z)&=a_0z^{-1}+\sum_{n=0}^{\infty}a_{p+n}z^{p+n}\in\Omega^*_{p0}(\alpha,\beta,\mu,z_0),\\ g(z)&=b_0z^{-1}+\sum_{n=0}^{\infty}b_{p+n}z^{p+n}\in\Omega^*_{p0}(\alpha,\beta,\mu,z_1),\end{aligned}$$

and $0\leq\lambda\leq 1$, that there exists a $z_2(z_0\leq z_2\leq z_1)$ such that

$$h(z)=\lambda f(z)+(1-\lambda)g(z)$$

is in the $\Omega^*_{p0}(\alpha,\beta,\mu,z_2)$. Since $f\in\Omega^*_{p0}(\alpha,\beta,\mu,z_0)$ and $g(z)\in\Omega^*_{p0}(\alpha,\beta,\mu,z_1)$. We have

$$\begin{aligned} a_0 &= 1-\sum_{n=0}^{\infty} a_{p+n} z_0^{p+n+1} \\ b_0 &= 1-\sum_{n=0}^{\infty} b_{p+n} z_1^{p+n+1}. \end{aligned}$$

Therefore we have

$$\begin{aligned} t(z) &= zh(z) \\ &= \lambda a_0 + (1-\lambda)b_0 + \lambda\sum_{n=0}^{\infty} a_{p+n} z^{p+n} + (1-\lambda)\sum_{n=0}^{\infty} b_{p+n} z^{p+n} \\ &= 1+\lambda\sum_{n=0}^{\infty}(z^{p+n}-z_0^{p+n+1})a_{p+n} + (1-\lambda)\sum_{n=0}^{\infty}(z^{p+n+1}-z_1^{p+n+1})b_{p+n} \quad (6.1) \end{aligned}$$

$t(z)$ being real when z is real with $t(z_0)\le 1$ and $t(z_1)\ge 1$, there exists $z_2\in[z_0,z_1]$, such that $t(z_2)=1$. This implies that

$$z_2 h(z_2) = 1 \text{ for some } z_2, z_0\le z_2\le z_1, \text{ that is } h(z)\in\Omega_{p0}.$$

Now, in view of (6.1) and $z_2h(z_2) = 1$, we have

$$\sum_{n=0}^{\infty}[(p+n)(1+\mu\beta) - \beta(1-\alpha)(1+\mu)z_2^{p+n+1}]\{\lambda a_{p+n} + (1-\lambda)b_{p+n}\}$$

$$= \lambda\sum_{n=0}^{\infty}[(p+n)(1+\mu\beta) - \beta(1-\alpha)(1+\mu)z_0^{p+n+1}]a_{p+n}$$

$$+(1-\lambda)\sum_{n=0}^{\infty}[(p+n)(1+\mu\beta) - \beta(1-\alpha)(1+\mu)z_1^{p+n+1}]b_{p+n}$$

$$+\beta(1-\alpha)(1+\mu)\lambda\sum_{n=0}^{\infty}[z_2^{p+n+1} - z_0^{p+n+1}]a_{p+n}$$

$$+\beta(1-\alpha)(1+\mu)(1-\lambda)\sum_{n=0}^{\infty}[z_2^{p+n+1} - z_1^{p+n+1}]b_{p+n}$$

$$= \lambda\sum_{n=0}^{\infty}[(p+n)(1+\mu\beta) + \beta(1-\alpha)(1+\mu)z_0^{p+n+1}]a_{p+n}$$

$$+(1-\lambda)\sum_{n=0}^{\infty}[(p+n)(1+\mu\beta) + \beta(1-\alpha)(1+\mu)z_1^{p+n+1}]b_{n+p}$$

$$\leq \lambda\beta(1-\alpha)(1+\mu) + (1-\lambda)\beta(1-\alpha)(1+\mu)$$

$$= \beta(1-\alpha)(1+\mu)$$

by Theorem 2, since $f(z) \in \Omega_{p0}^*(\alpha, \beta, \mu, z_0)$ and $g(z) \in \Omega_{p0}^*(\alpha, \beta, \mu, z_1)$. Hence we have $h(z) \in \Omega_{p0}^*(\alpha, \beta, \mu, z_2)$, by Theorem 2. Since z_0, z_1 and z_2 are arbitrary, the family $\Omega_{p0}^*(\alpha, \beta, \mu, B)$ is convex.

Conversely, if B is not connected, then there exists z_0, z_1 and z_2 such that $z_0, z_1 \in B$ and $z_2 \notin B$ and $z_0 < z_2 < z_1$. Assume that $f(z) \in \Omega_{p0}^*(\alpha, \beta, \mu, z_0)$ and $g(z) \in \Omega_{p0}^*(\alpha, \beta, \mu, z_1)$ are not both equal to $1/z$. Then, for fixed z_2 and $0 \leq \lambda \leq 1$, we have from (6.1)

$$t(\lambda) = t(z_2, \lambda) = 1 + \lambda\sum_{n=0}^{\infty}a_{p+n}(z_2^{p+n+1} - z_0^{p+n+1}) + (1-\lambda)\sum_{n=0}^{\infty}b_{p+n}(z_2^{p+n+1} - z_1^{p+n+1}).$$

Since $t(z_2, 0) < 1$ and $t(z_2, 1) > 1$, there must exists; $\lambda_0, 0 < \lambda_0 < 1$, such that $t(z_2, \lambda_0) = 1$ or $z_2h(z_2) = 1$, where $h(z) = \lambda_0 f(z) + (1-\lambda_0)g(z)$. Thus $h(z) \in \Omega_{p0}^*(\alpha, \beta, \mu, z_2)$. From Lemma 1, we have $h(z) \notin \Omega_{p0}^*(\alpha, \beta, \mu, B)$. Since $z_2 \in B$ and $h(z) \neq z$. This implies that the family $\Omega_{p0}^*(\alpha, \beta, \mu, B)$ is not convex which is a contradiction.

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