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ON SEPARABLE EXTENSIONS OF GROUP GRADED ALGEBRAS

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Abstract. We study extension $A \to B$ of G-graded $\mathcal{O}$-algebra, when G is a finite group. Such extensions occur when we consider blocks of normal subgroups and the associated graded source algebra, and we prove a refinement of a lifting theorem by B. Külshammer, T. Okuyama and A. Watanabe.

1. G-graded interior algebras

1.1. Let G be a finite group and let $\mathcal{O}$ be a complete discrete valuation ring with residue field $k = \mathcal{O}/J(\mathcal{O})$ of characteristic $p > 0$. Let $A = \bigoplus_{g\in G} A_g$ and $B = \bigoplus_{g\in G} B_g$ be two G-graded $\mathcal{O}$-algebras, and recall that the $\mathcal{O}$-algebra homomorphism $f\colon A \to B$ is called G-graded if $f(A_g) \subseteq B_g$ for all $g \in G$.

If $M = \bigoplus_{x\in G} M_x$ is a G-graded (A, A)-bimodule, we shall consider the $\mathcal{O}$-submodule

$$M^A = \{m \in M \mid am = ma \text{ for all } a \in A\}.$$

We say that B is a G-graded A-interior $\mathcal{O}$-algebra if B is a G-graded (A, A)-bimodule and $(xa)y = x(ay)$ for all $a \in A$, $x, y \in B$. Observe that in this case, the map $\beta : A \to B$, $\beta(a) = a1_B = 1_B a$ is a unitary homomorphism of G-graded $\mathcal{O}$-algebras. Conversely, given a unitary homomorphism $\beta : A \to B$ of G-graded $\mathcal{O}$-algebras, then B becomes a G-graded A-interior $\mathcal{O}$-algebra in an obvious way.

Let B be a G-graded A-interior $\mathcal{O}$-algebra. It is well-known that $B \otimes_A B$ is a G-graded (B, B)-bimodule, where if $x \in B_g$ and $y \in B_h$ then, by definition $x \otimes_A y \in (B \otimes_A B)_{gh}$. Remark that the multiplication map

$$\mu\colon B \otimes_A B \to B, \qquad \mu(x \otimes_A y) = xy$$

for $x, y \in B$ is a homomorphism of G-graded (B, B)-bimodules.

Lemma 1.2. *Let B be a G-graded A-interior $\mathcal{O}$-algebra and let $\mu\colon B\otimes_A B\to B$ denote the map of G-graded (B,B)-bimodules satisfying $\mu(x\otimes y)=xy$ for $x,y\in G$. Then the following statements are equivalent:*

(1) There exists a homomorphism $\nu: B\to B\otimes_A B$ of G-graded (B,B)-bimodules such that $\mu\circ\nu=1_B$.

(2) There exists an element $w=\sum_{j=1}^k x_j\otimes y_j\in(B\otimes_A B)^B$ where x_j,y_j are homogeneous elements such that $\sum_{j=1}^k x_jy_j=1_B$.

Proof. If (1) holds, let $\nu\colon B\to B\otimes_A B$ be a map of G-graded (B,B)-bimodules such that $\mu\circ\nu=\mathrm{id_B}$, and let $w=\nu(1_B)$.

Then $w\in(B\otimes_A B)^B$ and $\mu(w)=\mu(\nu(1_B))=1_B$. Since w has the form $w=\sum_{j=1}^k x_j\otimes y_j$ and each x_j, y_j is a sum of homogeneous elements, we may clearly assume that x_j,y_j are homogeneous, so (2) holds.

Conversely, assume that (2) holds. Since $\mu(w)=\sum_{j=1}^k x_jy_j=1\in B_1$, it follows that if $x_j\in B_g$, then $y_j\in B_{g^{-1}}$, so $w\in(B\otimes_A B)_1$. Then the map

$$\nu\colon B\to B\otimes_A B,\quad \nu(x)=xw=wx$$

is a homomorphism of G-graded (B,B)-bimodules, as for $x_g\in B_g$, we have $\nu(x_g)=x_gw\in(B\otimes_A B)_g$. Moreover, for all $x\in B$ $\mu(\nu(x))=x\mu(\nu(1))=x\mu(w)=x1_B=x$.

1.3. We say B is a *separable* G-graded A-interior $\mathcal{O}$-algebra, if the equivalent condition of Lemma 1.2 are satisfied.

This discussion is motivated by the following situation considered in [3] and [4].

Let H be a finite group, N a normal subgroup of H and let $G=H/N$. Then the group algebra $\mathcal{O}H$ can be regarded as a G-graded algebra, and $\mathcal{O}N$ is also an H-algebra. Let $b\in Z(\mathcal{O}N)$ be a block independent, and assume that b is G-invariant, that is, $b\in Z(\mathcal{O}H)$. Then the algebra $B=b\mathcal{O}H$ is a strongly G-graded $\mathcal{O}$-algebra. Note that $\beta=\{b\}$ is a point of N on B_1 and $\alpha=\{b\}$ is a point of H on B_1.

Let P_γ be a defect pointed group of H_α. Recall that if

$$\mathrm{Br}_P^{B_1} : B_1^P \to B_1^P/(\sum_{Q<P} \mathrm{Tr} B_1^Q + J(\mathcal{O})B_1^P)$$

is the Brauer map, then there is a primitive idempotent $i \in B_1^P$ such that $Br_P^{B_1}(i) \neq 0$, and γ is the point of B_1^P containing i. The interior $\mathcal{O}P$-algebra $A := iBi = i\mathcal{O}Hi$ is called a *source algebra* of B. By [4, Proposition 3.2], A is a strongly G-graded algebra and the structural map $\mathcal{O}P \to A$, $u \mapsto iu = ui$ is a homomorphism of G-graded algebras in a natural way (the degree of $u \in P$ is $uN \in G$).

Lemma 1.4. *With the above notations, the algebra A is a separable G-graded $\mathcal{O}P$-interior $\mathcal{O}$-algebra.*

Proof. By [5, Lemma 14.1] there are elements $a, b \in B_1^P$ such that $1_B = \mathrm{Tr}_P^H(aib)$. Consider the element

$$v = \sum_{h\in[H/P]} hai \otimes ibh^{-1} \in \mathcal{O}H \otimes_{\mathcal{O}P} \mathcal{O}H,$$

where $[H/P]$ is a set of representatives for the left cosets of P in H. Then as in [2, Lemma 4], $vh = hv$ for all $h \in G$ and $\sum_{h\in[H/P]} haibt^{-1} = \mathrm{Tr}_P^H(aib) = 1_B$. It follows that the element $w = ivi \in (A \otimes_{\mathcal{O}P} A)^A$ satisfies $\mu(w) = i = 1_A$, hence by Lemma 1.2, A is a separable G-graded $\mathcal{O}P$-interior algebra.

2. **The lifting theorem**

The following result is a generalization to the case of G-graded algebras of [2, Theorem 3].

Theorem 2.1. *Let B be a separable G-graded A-interior $\mathcal{O}$-algebra, let I be an G-graded ideal in an arbitrary G-graded A-interior $\mathcal{O}$-algebra C such that $I \subseteq J(I)$, and let $\rho\colon B \to C/I$ be a unitary homomorphism of G-graded A-interior $\mathcal{O}$-algebras.*

Suppose that there exist a map of G-graded (A, A)-bimodules $\tau_0\colon B \to C$ such that $\tau_0(x) + I = \rho(x)$ for $x \in B$. Then there exists a homomorphism of G-graded A-interior $\mathcal{O}$-algebras $\tau\colon B \to C$ such that $\tau(x) + I = \rho(x)$ for $x \in B$.

Moreover τ is unitary and unique up to conjugation with elements in $1 + I_1^A$.

Proof. Since B is a separable G-graded A-interior $\mathcal{O}$-algebra there exists an element $w=\sum_{j=1}^{k} x_j\otimes y_j \in (B\otimes_A B)^B$ with $x_j, y_j\in B$ homogeneous elements such that $\sum_{j=1}^{k} x_j y_j = 1_B$.

The construction τ is given in [1, Theorem 3]. We only have to verify that τ is grade-preserving.

Consider the map of (A,A)-bimodules

$$\theta\colon B\otimes_A B\to I^{2^n},\quad \theta(x\otimes y)=\tau_n(xy)-\tau_n(x)\tau_n(y)$$

for $x,y\in B$. If $x_g\in B_g$ and $y_h\in B_h$ we have $x_g y_h\in B_{gh}$ because B is a G-graded A-algebra. We know that τ_n is a G-graded map of (A,A)-bimodules, so $\tau(x_g y_h)\in C_{gh}$ and $\tau(x_g)\in C_g, \tau(y_h)\in C_h$ (hence $\tau(x_g)\tau(y_h)\in C_g C_h\subseteq C_{gh}$). Finally, $\tau_n(x_g y_h)-\tau_n(x_g)\tau_n(y_h)\in C_{gh}$, so $\theta(x\otimes y)\in I^{2^n}\cap C_{gh}$. This means that θ G-graded.

Consider the map of (A,A)-bimodules

$$\lambda\colon B\otimes_A B\otimes_A B\to I^{2^n},\qquad \lambda(x\otimes y\otimes z)=\theta(x\otimes y)\tau_n(z).$$

If $x_g\in B_g$, $y_h\in B_h$ and $z_l\in B_l$ we have $x_g y_h\in B_{gh}$ and $z_l\in B_l$. Since θ and τ_n are G-graded, we have $\theta(x_g\otimes y_h)\in I^{2^n}\cap C_{gh}$ and $\tau_n(z_l)\in I^{2^n}\cap C_l$. It follows that $\theta(x_g\otimes y_h)\tau_n(z_l)$ belongs to $(I^{2^n}\cap C_{gh})(I^{2^n}\cap C_l)\subseteq I^{2^n}\cap C_{ghl}$, hence $\lambda(x_g\otimes y_h\otimes z_l)\in I^{2^n}\cap C_{ghl}$. Then

$$\eta\colon B\to I^{2^n},\quad \eta(x)=\lambda(x\otimes w)=\sum_{j=1}^{k}\theta(x\otimes x_j)\tau_n(y_j)$$

is a map of G-graded (A,A)-bimodules with

$$\tau_n(x)\eta(y)-\eta(xy)+\eta(x)\tau_n(y)+I^{2^{n+1}}=\tau_n(xy)-\tau_n(x)\tau_n(y)+I^{2^{n+1}}.$$

If $x_g\in B_g$ then $x_g\otimes w\in B\otimes_A B$, and since λ is G-graded, we obtain $\lambda(x_g\otimes w)\in I^{2^n}\cap C_g$. It follows that η is G-graded. We get a map of (A,A)-bimodules

$$\tau_{n+1}\colon B\to C,\quad \tau_{n+1}(x)=\tau_n(x)+\eta(x)$$

with $\tau_{n+1}(x) + I^{2^n} = \tau_n(x) + I^{2^n}$ and $\tau_{n+1}(x)\tau_{n+1}(y) + I^{2^{n+1}} = \tau_{n+1}(xy) + I^{2^{n+1}}$ for $x, y \in B$. Since τ_n and η are G-graded, if $x_g \in B_g$ we have $\tau_n(x_g) \in C_g$ and $\eta(x_g) \in I^{2^n} \cap C_g$, so $\tau_n(x_g) + \eta(x_g) \in C_g$. Consequently τ_{n+1} is G-graded.

We have constructed the sequence $(\tau_n)_{n=0}^{\infty}$. Since $I \subseteq J(C)$ the map

$$\tau\colon B \to C, \quad \tau(x) = \lim_{n\to\infty} \tau_n(x)$$

is a well-defined unitary homomorphism of G-graded A-interior $\mathcal{O}$-algebras such that $\tau(x) + I = \tau_0(x) + I = \rho(x)$ for $x \in B$.

Finally, suppose that $\tau' : B \to C$ is another homomorphism of G-graded A-interior $\mathcal{O}$-algebras such that $\tau'(x) + I = \rho(x)$ for $x \in B$. Then

$$\delta\colon B \to I, \quad \delta(x) = \tau(x) - \tau'(x)$$

is a map of G-graded (A, A)-bimodules such that $\delta(xy) = \tau(x)\delta(y) + \delta(x)\tau'(y)$ for $x, y \in B$ and clearly δ is grade-preserving.

We consider the map of (A, A)-bimodules

$$\Phi\colon B \otimes_A B \to I, \quad \Phi(x \otimes y) = \tau(x)\delta(y)$$

for $x, y \in B$ and let $a = \Phi(w) = \sum_{j=1}^{k} \tau(x_j)\delta(y_j) \in I_1^A$. If $x_g \in B_g$ and $y_h \in B_h$ then, since τ and δ are G-graded we have $\tau(x_g)\delta(y_h) \in C_g(I \cap C_h) \subseteq I \cap C_{gh}$, hence Φ is G-graded too. Because B is a separable G-graded A-interior $\mathcal{O}$-algebra, there exists an element $w = \sum_{j=1}^{k} x_j \otimes y_j \in (B \otimes_A B)^B$ where x_j, y_j are homogeneous elements such that $\sum_{j=1}^{k} x_j y_j = 1_B$ imply $w \in (B \otimes_A B)_1$. We have that Φ is a map of G-graded (A, A)-bimodules. Therefore, if $x_j \in B_g$ and $y_j \in B_{g^{-1}}$ then $\tau(x_j) \in C_g$ and $\delta(y_j) \in I \cap C_{g-1}$, so $\tau(x_j)\delta(y_j) \in I^A \cap C_1 = I_1^A$.

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