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SOME REMARKS ON GROUPS OF POINTWISE SYMMETRIES OF THIRD-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. A necessary and sufficient condition for a third-order ordinary differential equation to possess a five-dimensional group of pointwise symmetries is established.

1. Introduction

The investigation of symmetries groups of differential equations in general (and of ordinary differential equations in particular) is one of the most important problems of differential equations geometry. The author of the present article studies third-order ordinary differential equations. Before that [1] the author obtained a complete solution of the problem in the case when such an equation has a seven-dimensional or a six-dimensional group of pointwise symmetries: the corresponding criteria have been obtained. (We recall [2] that seven is the maximum of the possible dimension of the pointwise symmetries group of a third-order ordinary differential equation). The present work is devoted to the analysis of the problem in the case when the dimension of the pointwise symmetries group is equal to five.

2. Preliminaries

We consider a third-order ordinary differential equation

$$y''' = f(x, y, y', y'') \tag{1}$$

given on a plane where the pseudo-group of point analytical transformations of coordinates acts:

$$\tilde{x} = \varphi_1(x, y);\ \tilde{y} = \varphi_2(x, y).$$

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The equation (1) is bound in an invariant way (concerning the given transformations) with such a geometrical object as a fiber space with a connection. The Cartan structural equations of the above-mentioned fiber space looks as follows:

$$\begin{gathered} D\omega^1 = \omega^1 \wedge \omega_1^1 + \Omega^1 \\ D\omega^2 = \omega^1 \wedge \omega_1^2 + \omega^2 \wedge \omega_2^2 \\ D\omega_1^2 = \omega_1^2 \wedge (\omega_2^2 - \omega_1^1) + \omega^1 \wedge \omega_{11}^2 + \omega^2 \wedge \omega_{11}^1 \\ D\omega_{11}^2 = \omega_{11}^2 \wedge (\omega_2^2 - 2\omega_1^1) + \omega_1^2 \wedge \omega_{11}^1 + \Omega_{11}^2 \\ D\omega_1^1 = \omega^1 \wedge \omega_{11}^1 + \Omega_1^1 \\ D\omega_2^2 = \omega^1 \wedge \omega_{11}^1 + \Omega_2^2 \\ D\omega_{11}^1 = \omega_1^1 \wedge \omega_{11}^1 + \Omega_{11}^1. \end{gathered} \tag{2}$$

The torsion-curvature forms of the equations (2) looks as follows:

$$\begin{gathered} \Omega^1 = \frac{1}{2}(a\omega_1^2 + b\omega_{11}^2) \wedge \omega^2 \\ \Omega_{11}^2 = \frac{1}{2}(c\omega^1 - e\omega_1^2) \wedge \omega^2 \\ \Omega_1^1 = \frac{1}{2}(g\omega^1 + h\omega_1^2 + k\omega_{11}^2) \wedge \omega^2 + \frac{1}{2}b\omega_{11}^2 \wedge \omega_1^2 \\ \Omega_2^2 = \frac{1}{2}[3g\omega^1 + (3h - 2m)\omega_1^2 + (3k - 2a)\omega_{11}^2] \wedge \omega^2 + \frac{1}{2}b\omega_1^2 \wedge \omega_{11}^2 \\ \Omega_{11}^1 = (\frac{1}{2}e + g)\omega^1 \wedge \omega_1^2 + \frac{1}{2}[n\omega^1 + r\omega_1^2 + (h + m)\omega_{11}^2] \wedge \omega^2 + (\frac{1}{2}a - k)\omega_1^2 \wedge \omega_{11}^2. \end{gathered} \tag{3}$$

The coefficients

$$a, b, c, e, g, h, k, m, n, r \tag{4}$$

being present in the torsion-curvature forms make up a complete system of differential invariants of the equation (1). They completely characterize the equation (1)and, thus, determine its geometry. The differentials of the invariants are as follows:

$$\begin{gathered} da + 2a(\omega_1^1 - \omega_2^2) - b\omega_{11}^1 = h\omega^1 + ... \\ db + b(3\omega_1^1 - 2\omega_2^2) = (k - a)\omega^1 + ... \\ dc - 3c\omega_1^1 = \sigma_1 \\ de - e(\omega_1^1 + \omega_2^2) = \sigma_2 \\ dg - g(\omega_1^1 + \omega_2^2) = \sigma_3 \end{gathered} \tag{5}$$

$$dh + h(\omega_1^1 - 2\omega_2^2) + (a-k)\omega_{11}^1 = \sigma_4$$

$$dk + 2k(\omega_1^1 - \omega_2^2) = \sigma_5$$

$$dm + m(\omega_1^1 - 2\omega_2^2) = (r+bc)\omega^1 + ...$$

$$dn - n(2\omega_1^1 + \omega_2^2) - (g+e)\omega_{11}^1 = \sigma_6$$

$$dr - 2r\omega_2^2 - m\omega_{11}^1 = \sigma_7.$$

The right parts of all equalities are linear combinations of the main forms of the second-order tangent element: $\omega^1, \omega^2, \omega_1^2, \omega_{11}^2$. We denote such combinations by the symbols $\sigma_i, \sigma, \tilde{\sigma}$. From the given relations it is seen that the differential invariants of the equation (1) are either relative invariants or become relative invariants when some relative invariants vanish.

3. **The main result**

Now, we consider the invariant c. According to (5),

$$dc - 3c\omega_1^1 = \sigma_1. \tag{6}$$

Thus, c is one of the invariants that is relative from the beginning. For this reason for c, as well as for any relative invariant, two different cases are possible: $c = 0$ and $c \neq 0$. From the multitude of third-order ordinary differential equations we select those for which the invariant c is different from zero and all others differential invariants vanish:

$$a = b = e = g = h = k = m = n = r = 0. \tag{7}$$

Let us do the canonization $c \overset{k}{=} 1$. In extracted particular case according to (6), the differential form ω_1^1 will be a linear combination of the main forms of the second-order tangent element:

$$\omega_1^1 = t\omega^1 + t_1\omega^2 + t_2\omega_1^2 + t_3\omega_{11}^2, \tag{8}$$

where t, t_1, t_2, t_3 are some new invariants. Having an exterior differentiation of the equality (8), we shall find the relations for differentials of these invariants:

$$dt = \sigma_8;\ dt_1 - t_1\omega_2^2 = \sigma_9;$$

$$dt_2 - t_2\omega_2^2 = \sigma_{10};\ dt_3 - t_3\omega_2^2 = \sigma_{11}. \tag{9}$$

The obtained relations show that the coefficients $t_1, t_2\ \ t_3$ are relative invariants and t is an absolute invariant.

The exterior differentiation of (8) will give us another useful equality:

$$\omega^1_{11} = p\omega^1 + p_1\omega^2 + p_2\omega^2_1 + p_3\omega^2_{11}. \tag{10}$$

(Here $p, p_1, p_2\ \ p_3$ are new invariants of the equation (1) we are interested in). Having an exterior differentiation of (10), we get:

$$dp = \sigma_{12};\ dp_1 - p_1\omega^2_2 = \sigma_{13};$$

$$dp_2 - p_2\omega^2_2 = \sigma_{14};\ dp_3 - p_3\omega^2_2 = \sigma_{15}. \tag{11}$$

Therefore, p_1, p_2, p_3 are relative invariants and p is an absolute invariant.

Let us select the case when all the "new" relative invariants vanish:

$$t_1 = t_2 = t_3 = p_1 = p_2 = p_3 = 0. \tag{12}$$

In this case the equalities (8) and (10) will look as follows: $\omega^1_1 = t\omega^1$; $\omega^1_{11} = p\omega_1$, and the Cartan structural equations (2) will be written down as follows:

$$D\omega^1 = D\omega^2_2 = 0$$

$$D\omega^2 = \omega^1 \wedge \omega^2_1 + \omega^2 \wedge \omega^2_2$$

$$D\omega^2_1 = \omega^2_1 \wedge (\omega^2_2 - t\omega^1) + \omega^1 \wedge (\omega^2_{11} - p\omega^2) \tag{13}$$

$$D\omega^2_2 = \omega^2_{11} \wedge (\omega^2_2 - 2t\omega^1) + \omega^1 \wedge (\frac{1}{2}\omega^2 - p\omega^2_1).$$

Having an exterior differentiation of (13), we shall be convinced that (13) are structure equations of a some transformations group. The dimension of this group is equal to five. For the equation (1) that we are interested in the group mentioned is a group of pointwise symmetries (in the selected particular case). So, we have proved

Proposition I. *If $c \neq 0$ and equalities (7) and (12) are fulfilled, then the equation (1) has a five-dimensional group of pointwise symmetries. The structure equations of this group looks as (13).*

It turns out that the inverse statement is also true.

Proposition II. *If the equation (1) has a five-dimensional group of pointwise symmetries, then $c \neq 0$ and equalities (7) and (12) are fulfilled.*

Proof. Assume the equation (1) possesses a five-dimensional group of pointwise symmetries. Then among its differential invariants in (4) there is at least one that is different from zero. Otherwise [1], the symmetries group is the seven-dimensional group $g_{2,6}(3)$ (using the Cartan's terminology [3]).

Let I be one of the relative invariants of the equation (1). Then its differential satisfies the equality:

$$dI + I(s_1\omega_1^1 + s_2\omega_2^2) = r_1\omega^1 + r_2\omega^2 + r_3\omega_1^2 + r_4\omega_{11}^2. \tag{14}$$

We assume that $I \neq 0$. Canonizing $I \overset{k}{=} 1$, from (14) we obtain:

$$s_1\omega_1^1 + s_2\omega_2^2 = r_1\omega^1 + r_2\omega^2 + r_3\omega_1^2 + r_4\omega_{11}^2. \tag{15}$$

Having an exterior differentiation of this equality, as one of differential results we obtain the relation:

$$dr_1 - r_1\omega_1^1 + (s_1 + s_2)\omega_{11}^1 = \sigma.$$

The invariant r_1 (like the others invariants of the equation (1)) must be a constant in the case when (2) are the structure equations of the symmetries group of (1). Under this condition the last equality looks as follows:

$$-r_1\omega_1^1 + (s_1 + s_2)\omega_{11}^1 = \sigma. \tag{16}$$

Now, we assume that the invariant s_2 is different from zero and express the differential form ω_2^2. Substituting the relation for ω_2^2 in (2), we obtain:

$$D\omega^2 = \omega^1 \wedge \omega_1^2 + \omega^2 \wedge (-\frac{s_1}{s_2}\omega^1 + \tilde{\sigma}).$$

We have an exterior differentiation of this equality. Among others we obtain the following relation:

$$-\frac{s_1}{s_2} = 1, \text{ or } s_1 + s_2 = 0.$$

As seen from (5), among the relative invariants only the invariant k satisfies this condition. But being different from zero, the invariant k does not suit us for the reason that if we admit that the equation (1) has not any other invariants different from zero, then the symmetries group will be the six-dimensional group $g_{4,2}$. If at least one invariant is different from zero, then according to (15) and (16) the forms

$\omega_1^1, \omega_2^2, \omega_{11}^1$ will turn out to be dependent on $\omega^1, \omega^2, \omega_1^2, \omega_{11}^2$ and, thus, the symmetries group can not have a dimension more than four.

Hence, all the relative invariants from (4) for which $s_2 \neq 0$ must vanish. Therefore

$$b = e = g = k = m = 0 \Rightarrow a = k = 0.$$

Moreover, the coefficients $h, n,$ become relative invariants for which $s_2 \neq 0$. If we consider this fact as mentioned before, we can come to conclusion that $h = n = r = 0$.

Only the invariant $I = c$ satisfies the condition $s_2 = 0$. Therefore, $c \neq 0$.

In this case, as it is mentioned before, the forms ω_1^1 and ω_{11}^1 are expressed in a linear way though the main forms of the second order tangent element. If we admit that any of invariants $t_1, t_2, t_3, p_1, p_2, p_3,$ are different from zero, then owing to (9) and (11) the form ω_2^2 will also be dependent on $\omega^1, \omega^2, \omega_1^2, \omega_{11}^2$, and so the symmetries group can not have a dimension more then four.

That's why, in the case we are interested in $t_1 = t_2 = t_3 = p_1 = p_2 = p_3 = 0$. The Proposition II is proved completely.

Proposition III. *The structure equations (13) determine the transformations group* $g_{5,5}$.

Proof. We substitute

$$\omega^1 = \Theta^2;\ \omega^2 = \Theta^1;\ \omega_2^2 = \Theta_1^1 + t\Theta^2;$$

$$\omega_1^2 = \Theta_2^1 + t\Theta^1;\ \omega_{11}^2 = -\Theta_{22}^1 + t\Theta_2^1 + p\Theta^1.$$

According to the substitution, the equations (13) may be written down as follows:

$$\begin{gathered} D\Theta^2 = D\Theta_1^1 = 0 \\ D\Theta^1 = \Theta^1 \wedge \Theta_1^1 + \Theta^2 \wedge \Theta_2^1 \\ D\Theta_2^1 = \Theta_{22}^1 \wedge \Theta^2 + \Theta_2^1 \wedge \Theta_1^1 \\ D\Theta_{22}^1 = \Theta^2 \wedge ((2p - t^2)\Theta_2^1 - \frac{1}{2}\Theta^1) + \Theta_{22}^1 \wedge \Theta_1^1. \end{gathered} \tag{17}$$

Now, we use the structural equations of the group $g_{5,5}$ for the third-order ordinary differential equation [3]:

$$D\Theta^1 = \Theta^1 \wedge \Theta_1^1 + \Theta^2 \wedge \Theta_2^1$$

$$D\Theta^2 = 0$$

$$D\Theta^1_1 = 0$$

$$D\Theta^1_2 = \Theta^1_{22} \wedge \Theta^2 + \Theta^1_2 \wedge \Theta^1_1$$

$$D\Theta^1_{22} = \Theta^2 \wedge (m_2\Theta^1_2 + m_3\Theta^1) + \Theta^1_{22} \wedge \Theta^1_1.$$

It is quite evident that (17) are structure equations of $g_{5,5}$, in addition, $m_2 = 2p - t^2; m_3 = -\frac{1}{2}$. The Proposition III have been proved.

Remark. In our case the finite transformations of the group look as follows:

$$\tilde{x} = c_1 x + \psi(y);$$

$$\tilde{y} = y + c_2,$$

where $\psi(y)$ is the general solution of the equation

$$\psi''' - (t^2 - 2p)\psi' + \frac{1}{2}\psi = 0.$$

Taking together all the proved statements, we state the following result:

Theorem. *Third-order ordinary differential equations have a five-dimensional group of pointwise symmetries if and only if* $c \neq 0$*, and conditions (7) and (12) are fulfilled. In addition, the only possible group of pointwise symmetries is (with the precision to an isomorphism) the group* $g_{5,5}$.

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