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ON CERTAIN INEQUALITIES INVOLVING THE IDENTRIC MEAN IN n VARIABLES

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Abstract. In this paper we establish one Chebyshev type and two Ky Fan type inequalities for the weighted identric mean in n variables.

1. Introduction and notation

Let $n \geq 2$ be a given integer, let

$$A_{n-1} = \{ (\lambda_1, \dots, \lambda_{n-1}) \mid \lambda_i \geq 0,\ i = 1, \dots, n-1,\ \lambda_1 + \cdots + \lambda_{n-1} \leq 1 \}$$

be the Euclidean simplex, and let μ be a probability measure on A_{n-1}. For each $i \in \{ 1, \dots, n \}$, the ith weight w_i associated to μ is defined by

$$\begin{aligned} w_i &= \int_{A_{n-1}} \lambda_i d\mu(\lambda) \qquad \text{if} \quad 1 \leq i \leq n-1, \\ w_n &= \int_{A_{n-1}} (1 - \lambda_1 - \cdots - \lambda_{n-1}) d\mu(\lambda), \end{aligned}$$

where $\lambda = (\lambda_1, \dots, \lambda_{n-1}) \in A_{n-1}$. Obviously, $w_i > 0$ for all $i \in \{ 1, \dots, n \}$, and $w_1 + \cdots + w_n = 1$. We also define

$$\begin{aligned} w_{ij} &= \int_{A_{n-1}} \lambda_i \lambda_j d\mu(\lambda) \qquad \text{if} \quad 1 \leq i, j \leq n-1, \\ w_{in} &= w_{ni} = \int_{A_{n-1}} \lambda_i (1 - \lambda_1 - \cdots - \lambda_{n-1}) d\mu(\lambda) \qquad \text{if} \quad 1 \leq i \leq n-1, \\ w_{nn} &= \int_{A_{n-1}} (1 - \lambda_1 - \cdots - \lambda_{n-1})^2 d\mu(\lambda). \end{aligned}$$

Taking into account the Liouville formula (see, for instance, [1])

$$\begin{aligned} &\int_{A_{n-1}} \lambda_1^{p_1 - 1} \cdots \lambda_{n-1}^{p_{n-1} - 1} f(\lambda_1 + \cdots + \lambda_{n-1}) d\lambda_1 \cdots d\lambda_{n-1} \\ &\qquad = \frac{\Gamma(p_1) \cdots \Gamma(p_{n-1})}{\Gamma(p_1 + \cdots + p_{n-1})} \int_0^1 x^{p_1 + \cdots + p_{n-1} - 1} f(x) dx, \end{aligned}$$

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in the special case $\mu = (n-1)!$ we get $w_i = 1/n$ for all $i \in \{1, \ldots, n\}$ and

$$\begin{aligned} w_{ii} &= \frac{2}{n(n+1)} \qquad \text{for all} \quad i \in \{1, \ldots, n\}, \\ w_{ij} &= \frac{1}{n(n+1)} \qquad \text{for all} \quad i, j \in \{1, \ldots, n\}, \ i \neq j. \end{aligned}$$

Next, recall that the *identric mean* $I(x_1, x_2)$ of the positive real numbers x_1 and x_2 is defined by

$$\begin{aligned} &I(x_1, x_2) = \frac{1}{e}\left(\frac{x_2^{x_2}}{x_1^{x_1}}\right)^{1/(x_2-x_1)} \qquad \text{if} \quad x_1 \neq x_2, \\ &I(x_1, x_1) = x_1. \end{aligned}$$

It is easily seen that the following integral representation holds:

$$I(x_1, x_2) = \exp\left(\int_0^1 \log(tx_1 + (1-t)x_2)dt\right). \tag{1.1}$$

Given $X = (x_1, \ldots, x_n) \in \,]0, \infty[^n$, we set

$$\lambda \cdot X := \lambda_1 x_1 + \cdots + \lambda_{n-1} x_{n-1} + (1 - \lambda_1 - \cdots - \lambda_{n-1}) x_n$$

for all $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in A_{n-1}$. Starting from (1.1), in [7] it was pointed out that

$$I(X; \mu) := \exp\left(\int_{A_{n-1}} \log(\lambda \cdot X) d\mu(\lambda)\right)$$

can be considered as the weighted identric mean of $x_1, \ldots, x_n$. For $\mu = (n-1)!$ we obtain the unweighted and symmetric identric mean of $x_1, \ldots, x_n$

$$I(X) = I(x_1, \ldots, x_n) = \exp\left((n-1)! \int_{A_{n-1}} \log(\lambda \cdot X) d\lambda_1 \cdots d\lambda_{n-1}\right).$$

As in the case of other means, $I(X; \mu)$ can be generalized as follows: for each real number r we set $X^r := (x_1^r, \ldots, x_n^r)$, and then define

$$\begin{aligned} I_r(X; \mu) &:= (I(X^r; \mu))^{1/r} \qquad \text{if} \quad r \neq 0, \\ I_0(X; \mu) &:= \lim_{r \to 0} I_r(X; \mu) = x_1^{w_1} \cdots x_n^{w_n} \qquad \text{(see [5])}. \end{aligned}$$

The means $I_r(X; \mu)$ are special cases of the so-called Stolarsky-Tobey means introduced in [5]: namely $I_r(X; \mu) = E_{r,r}(X; \mu)$. Consequently, several inequalities (of the Jensen, Minkowski, Hölder, Rennie, and Kantorovich type, respectively) involving the means I_r can be obtained as special cases of the results listed in [5]. In Section 2 of this paper we complete these inequalities by proving a Chebyshev type inequality for I_r.

Let

$$A(X;\mu) := w_1x_1 + \cdots + w_nx_n \qquad \text{and} \qquad G(X;\mu) := x_1^{w_1}\cdots x_n^{w_n}$$

be the weighted arithmetic and geometric mean, respectively, of $x_1, \ldots, x_n$. For $\mu = (n-1)!$ we obtain the usual arithmetic and geometric mean of $x_1, \ldots, x_n$

$$\begin{aligned} A(X) &= A(x_1,\ldots,x_n) = \frac{x_1+\cdots+x_n}{n}, \\ G(X) &= G(x_1,\ldots,x_n) = (x_1\cdots x_n)^{1/n}. \end{aligned}$$

A famous result due to Ky Fan asserts that if $0 < x_i \le 1/2$ for all $i \in \{1,\ldots,n\}$, then

$$\frac{G(X;\mu)}{G(\mathbf{1}-X;\mu)} \le \frac{A(X;\mu)}{A(\mathbf{1}-X;\mu)}, \tag{1.2}$$

where $\mathbf{1} - X := (1-x_1,\ldots,1-x_n)$. The following refinement of (1.2) has been recently obtained in [7]:

$$\frac{G(X;\mu)}{G(\mathbf{1}-X;\mu)} \le \frac{I(X;\mu)}{I(\mathbf{1}-X;\mu)} \le \frac{A(X;\mu)}{A(\mathbf{1}-X;\mu)}. \tag{1.3}$$

In Section 3 of this paper we establish a converse of the left inequality in (1.3) as well as an improvement of the right inequality in (1.3).

2. Chebyshev's inequality for the identric mean in n variables

Theorem 2.1. *Let $X = (x_1,\ldots,x_n) \in \mathbf{R}^n$ and $Y = (y_1,\ldots,y_n) \in \mathbf{R}^n$ such that $0 < x_1 \le \cdots \le x_n$ and $0 < y_1 \le \cdots \le y_n$, and let $X \cdot Y := (x_1y_1,\ldots,x_ny_n)$. Then*

$$\begin{aligned} I_r(X;\mu)I_r(Y;\mu) &\le I_r(X\cdot Y;\mu) \qquad \textit{for all} \quad r>0, \\ I_r(X;\mu)I_r(Y;\mu) &\ge I_r(X\cdot Y;\mu) \qquad \textit{for all} \quad r<0. \end{aligned}$$

Proof. According to Chebyshev's inequality, we have

$$(\lambda\cdot X^r)(\lambda\cdot Y^r) \le \lambda\cdot(X\cdot Y)^r$$

for all $r \in \mathbf{R}$ and all $\lambda \in A_{n-1}$, hence

$$\int_{A_{n-1}} \log(\lambda\cdot X^r)d\mu(\lambda) + \int_{A_{n-1}} \log(\lambda\cdot Y^r)d\mu(\lambda) \le \int_{A_{n-1}} \log(\lambda\cdot(X\cdot Y)^r)d\mu(\lambda)$$

for all $r \in \mathbf{R}$. Exponentiating both sides yields

$$I(X^r;\mu)I(Y^r;\mu) \le I((X\cdot Y)^r;\mu) \qquad \text{for all} \quad r \in \mathbf{R}.$$

This inequality implies the conclusion of the theorem. □

Besides the identric mean $I(x_1,x_2)$ of the positive real numbers x_1 and x_2, the logarithmic mean of x_1 and x_2 is another important special case of the Stolarsky mean of x_1 and x_2. Recall that the *logarithmic mean* of x_1 and x_2 is defined by

$$L(x_1,x_2) = \frac{x_1 - x_2}{\log x_1 - \log x_2} \qquad \text{if} \quad x_1 \ne x_2,$$
$$L(x_1,x_1) = x_1.$$

Theorem 2.2. *Let x_1, x_2, y_1, y_2 be positive real numbers.*
If $(x_1 - x_2)(y_1 - y_2) > 0$, then

$$L(x_1,x_2)L(y_1,y_2) < L(x_1y_1, x_2y_2), \tag{2.1}$$

while if $(x_1 - x_2)(y_1 - y_2) < 0$, then

$$L(x_1,x_2)L(y_1,y_2) > L(x_1y_1, x_2y_2). \tag{2.2}$$

In the proof we shall use the elementary

Lemma 2.3. *The following assertions are true:*

a) $f_1(v) = v\log v - v + 1$ is strictly decreasing from $]0,1[$ onto $]0,1[$, and strictly increasing from $]1,\infty[$ onto $]0,\infty[$.

b) $f_2(v) = v\log v - 2v + \log v + 2$ is strictly increasing from $]0,\infty[$ onto $]-\infty,\infty[$.

c) $f_3(v) = v^2 - 2v\log v - 1$ is strictly increasing from $]0,1[$ onto $]-1,0[$.

d) $f_4(v) = v\log^2 v - (v-1)^2$ is strictly increasing from $]0,1[$ onto $]-1,0[$.

Proof of the Theorem 2.2. Suppose first that $(x_1 - x_2)(y_1 - y_2) > 0$. Due to the symmetry, we may assume that $x_1 > x_2$ and $y_1 > y_2$, so $u := \frac{x_1}{x_2} > 1$, $v := \frac{y_1}{y_2} > 1$. Taking into account the homogeneity of L, inequality (2.1) is equivalent to

$$\frac{u-1}{\log u} \cdot \frac{v-1}{\log v} < \frac{uv-1}{\log u + \log v},$$

i. e. to

$$(u-1)(v-1)(\log u+\log v)-(uv-1)\log u\log v<0. \tag{2.3}$$

Let $v\in\,]1,\infty[$ be fixed, and let $f:]0,\infty[\to\mathbf{R}$ be the function defined by

$$f(u):=(u-1)(v-1)(\log u+\log v)-(uv-1)\log u\log v. \tag{2.4}$$

Then we have

$$f'(u)=(v-1-v\log v)\log u+\frac{u-1}{u}(v-1-\log v),$$
$$f''(u)=\frac{v-1-\log v-u(v\log v-v+1)}{u^2}.$$

Since $v>1$, by virtue of Lemma 2.3 a) and b) we obtain

$$f''(u)<\frac{v-1-\log v-(v\log v-v+1)}{u^2}=-\frac{v\log v-2v+\log v+2}{u^2}<0$$

for all $u\in\,]1,\infty[$, hence f' must be strictly decreasing on $]1,\infty[$. Therefore $f'(u)<0$ for $u>1$, because $f'(1)=0$. This implies that f is also strictly decreasing on $]1,\infty[$. Consequently, $f(u)<0$ for $u>1$, because $f(1)=0$. This proves the validity of (2.3).

Suppose now that $(x_1-x_2)(y_1-y_2)<0$, and assume that $x_1>x_2$ and $y_1<y_2$. Then we have $u:=\frac{x_1}{x_2}>1$ and $v:=\frac{y_1}{y_2}<1$. Depending on u and v, we distinguish the following possible cases:

Case I. $uv=1$.

Then inequality (2.2) is equivalent to $L(u,1)L(1/u,1)>1$. Since $L(1/u,1)=L(u,1)/u$, this transforms into the well-known inequality $L(u,1)>\sqrt{u}=G(u,1)$ (see [8]).

Case II. $uv>1$.

Then inequality (2.2) is equivalent to (2.3). Let $v\in\,]0,1[$ be fixed, and let $f:]0,\infty[\to\mathbf{R}$ be the function defined by (2.4). By virtue of Lemma 2.3 a) and c), for all $u\in\,]1/v,\infty[$ we have

$$f''(u)<\frac{v-1-\log v-\frac{1}{v}(v\log v-v+1)}{u^2}=\frac{v^2-2v\log v-1}{u^2v}<0,$$

hence f' must be strictly decreasing on $]1/v,\infty[$. But $f'(1/v)=v\log^2 v-(v-1)^2<0$, according to Lemma 2.3 d), so $f'(u)<0$ for $u>1/v$. This implies that f is also strictly decreasing on $]1/v, \infty[$. Consequently, $f(u) < 0$ for $u > 1/v$, because $f(1/v) = 0$. This proves the validity of (2.3).

Case III. $uv < 1$.

Then inequality (2.2) is equivalent to

$$(u-1)(v-1)(\log u + \log v) - (uv-1)\log u \log v > 0. \tag{2.5}$$

Let again $v \in]0,1[$ be fixed, and let $f :]0,\infty[\to \mathbf{R}$ be the function defined by (2.4). Set

$$\tilde{v} := \frac{v - 1 - \log v}{v \log v - v + 1}.$$

By Lemma 2.3 a), b), and c) we have $1 < \tilde{v} < 1/v$. It is immediately seen that $f''(u) > 0$ for $u \in]1, \tilde{v}[$ and $f''(u) < 0$ for $u \in]\tilde{v}, 1/v[$. Consequently, f' is strictly increasing on $]1, \tilde{v}[$ and strictly decreasing on $]\tilde{v}, 1/v[$. Since $f'(1) = 0$ and $f'(1/v) = v \log^2 v - (v-1)^2 < 0$, it follows that there exists a unique $\bar{v} \in]\tilde{v}, 1/v[$ such that $f'(\bar{v}) = 0$, $f'(u) > 0$ for $u \in]1, \bar{v}[$, and $f'(u) < 0$ for $u \in]\bar{v}, 1/v[$. Therefore f is strictly increasing on $]1, \bar{v}[$ and strictly decreasing on $]\bar{v}, 1/v[$. Since $f(1) = f(1/v) = 0$, we can conclude that $f(u) > 0$ for all $u \in]1, 1/v[$. This completes the proof of (2.5). □

Remark. It would be interesting to study whether Theorem 2.2 can be generalized for n variables (the author does not know the answer).

3. Two inequalities related to (1.3)

In this section, both a converse of the left inequality in (1.3) and a refinement of the right inequality in (1.3) are obtained. They are contained in the following two theorems.

Theorem 3.1. *If $X = (x_1, \ldots, x_n) \in]0, 1/2]^n$, then it holds that*

$$\log \frac{I(X;\mu)}{I(\mathbf{1}-X;\mu)} - \log \frac{G(X;\mu)}{G(\mathbf{1}-X;\mu)} \tag{3.1}$$
$$\leq \left(\sum_{i=1}^{n} w_i x_i\right)\left(\sum_{i=1}^{n} \frac{w_i}{x_i(1-x_i)}\right) - \sum_{i=1}^{n} \frac{w_i}{1-x_i}.$$

Theorem 3.2. *If $X = (x_1, \dots, x_n) \in \,]0, 1/2]^n$, then it holds that*

$$\log \frac{A(X;\mu)}{A(\mathbf{1} - X;\mu)} - \log \frac{I(X;\mu)}{I(\mathbf{1} - X;\mu)} \tag{3.2}$$
$$\geq \frac{1 - 2\bar{x}}{2\bar{x}^2(1 - \bar{x})^2} \sum_{i,j=1}^{n} (w_{ij} - w_i w_j) x_i x_j,$$

where $\bar{x} := \max\{\, x_1, \dots, x_n \,\}$.

In the proofs of Theorem 3.1 and Theorem 3.2 we shall use the following lemmas.

Lemma 3.3. *Let $J \subseteq \mathbf{R}$ be a nonempty interval, let $X = (x_1, \dots, x_n) \in J^n$, and let $\phi : J \to \mathbf{R}$ be a twice differentiable function such that $\phi''(x) \geq 0$ for all $x \in J$. Then it holds that*

$$\sum_{i=1}^{n} w_i \phi(x_i) - \int_{A_{n-1}} \phi(\lambda \cdot X) d\mu(\lambda) \tag{3.3}$$
$$\leq \sum_{i=1}^{n} w_i x_i \phi'(x_i) - \left(\sum_{i=1}^{n} w_i x_i \right) \left(\sum_{i=1}^{n} w_i \phi'(x_i) \right).$$

Proof. The nonnegativity of ϕ'' ensures that

$$\phi(\lambda \cdot X) \geq \phi(x_i) + \phi'(x_i)(\lambda \cdot X - x_i)$$

for all $i \in \{\, 1, \dots, n \,\}$ and all $\lambda \in A_{n-1}$. Integrating over A_{n-1} with respect to μ yields

$$\phi(x_i) - \int_{A_{n-1}} \phi(\lambda \cdot X) d\mu(\lambda) \leq x_i \phi'(x_i) - \phi'(x_i)(w_1 x_1 + \dots + w_n x_n)$$

for all $i \in \{\, 1, \dots, n \,\}$. Multiplying both sides by w_i and then summing the obtained inequalities, we get (3.3). □

Given the nonempty interval $J \subseteq \mathbf{R}$, to each function $\phi : J \to \mathbf{R}$ we associate the function $L\phi : J^n \to \mathbf{R}$ defined by

$$L\phi(X) := \int_{A_{n-1}} \phi(\lambda \cdot X) d\mu(\lambda) - \phi\left(\sum_{i=1}^{n} w_i x_i \right) \qquad X = (x_1, \dots, x_n) \in J^n.$$

Lemma 3.4. *Suppose that ϕ has a continuous second derivative in J, and let $X = (x_1, \dots, x_n) \in J^n$, $\underline{x} := \min\{\, x_1, \dots, x_n \,\}$, $\bar{x} := \max\{\, x_1, \dots, x_n \,\}$. Then there* *exists a point* $\tilde{x} \in [\underline{x}, \bar{x}]$ *such that*

$$L\phi(X) = \frac{1}{2}\phi''(\tilde{x}) Le_2(X),$$

where $e_2(x) = x^2$.

Proof. Set $\lambda^0 := (w_1, \ldots, w_{n-1}) \in A_{n-1}$ and $x_0 := w_1 x_1 + \cdots + w_n x_n$. Obviously, $x_0 = \lambda^0 \cdot X$. Next, let $\varphi : A_{n-1} \to \mathbf{R}$ be the function defined by $\varphi(\lambda) := \phi(\lambda \cdot X)$. For each $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in A_{n-1}$ there exists $\xi \in]0, 1[$ such that

$$\varphi(\lambda) = \varphi(\lambda^0) + d\varphi(\lambda^0)(\lambda - \lambda^0) + \frac{1}{2} d^2\varphi(\lambda^0 + \xi(\lambda - \lambda^0))(\lambda - \lambda^0),$$

hence

$$\phi(\lambda \cdot X) = \phi(x_0) + \phi'(x_0) \sum_{i=1}^{n-1} (x_i - x_n)(\lambda_i - w_i) \tag{3.4}$$

$$+\frac{1}{2}\phi''(x_\xi) \sum_{i,j=1}^{n-1} (x_i - x_n)(x_j - x_n)(\lambda_i - w_i)(\lambda_j - w_j),$$

where $x_\xi := (\lambda^0 + \xi(\lambda - \lambda^0)) \cdot X$. Further, let

$$m := \inf \phi''([\underline{x}, \bar{x}]) \qquad \text{and} \qquad M := \sup \phi''([\underline{x}, \bar{x}]).$$

Taking into account that

$$\sum_{i,j=1}^{n-1} (x_i - x_n)(x_j - x_n)(\lambda_i - w_i)(\lambda_j - w_j) = \left(\sum_{i=1}^{n-1} (x_i - x_n)(\lambda_i - w_i) \right)^2 \geq 0,$$

from (3.4) we get

$$\frac{1}{2} m \sum_{i,j=1}^{n-1} (x_i - x_n)(x_j - x_n)(\lambda_i - w_i)(\lambda_j - w_j)$$

$$\leq \phi(\lambda \cdot X) - \phi(x_0) - \phi'(x_0) \sum_{i=1}^{n-1} (x_i - x_n)(\lambda_i - w_i)$$

$$\leq \frac{1}{2} M \sum_{i,j=1}^{n-1} (x_i - x_n)(x_j - x_n)(\lambda_i - w_i)(\lambda_j - w_j)$$

for all $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in A_{n-1}$. Integrating over A_{n-1} with respect to μ yields

$$\frac{1}{2} m \sum_{i,j=1}^{n-1} (w_{ij} - w_i w_j)(x_i - x_n)(x_j - x_n) \leq L\phi(X)$$

$$\leq \frac{1}{2} M \sum_{i,j=1}^{n-1} (w_{ij} - w_i w_j)(x_i - x_n)(x_j - x_n).$$

As a simple computation shows, we have

$$\sum_{i,j=1}^{n-1}(w_{ij}-w_iw_j)(x_i-x_n)(x_j-x_n)=Le_2(X),$$

hence $\frac{1}{2}mLe_2(X)\le L\phi(X)\le\frac{1}{2}MLe_2(X)$. Now, the continuity of ϕ'' ensures the existence of a point $\tilde{x}\in[\underline{x},\bar{x}]$ such that $L\phi(X)=\frac{1}{2}\phi''(\tilde{x})Le_2(X)$. □

Proof of the Theorem 3.1. Inequality (3.1) follows at once from (3.3) if we take $J:=]0,1/2]$ and $\phi:J\to\mathbf{R}$ to be the function $\phi(x)=\log(1-x)-\log x$, whose second derivative is

$$\phi''(x)=\frac{1-2x}{x^2(1-x)^2}\ge 0\qquad\text{for all } x\in J.$$

□

Proof of the Theorem 3.2. With the same choices for J and ϕ, from Lemma 3.4 we conclude the existence of a point $\tilde{x}\in[\underline{x},\bar{x}]$ such that

$$\begin{aligned}\log\frac{A(X;\mu)}{A(\mathbf{1}-X;\mu)}-\log\frac{I(X;\mu)}{I(\mathbf{1}-X;\mu)}&=\frac{1-2\tilde{x}}{2\tilde{x}^2(1-\tilde{x})^2}Le_2(X)\\&=\frac{1-2\tilde{x}}{2\tilde{x}^2(1-\tilde{x})^2}\sum_{i,j=1}^{n}(w_{ij}-w_iw_j)x_ix_j\\&\ge\frac{1-2\bar{x}}{2\bar{x}^2(1-\bar{x})^2}\sum_{i,j=1}^{n}(w_{ij}-w_iw_j)x_ix_j,\end{aligned}$$

because ϕ'' is decreasing on J. □

Remark. For $\mu=(n-1)!$, inequalities (3.1) and (3.2) reduce to

$$\log\frac{I(X)}{I(\mathbf{1}-X)}-\log\frac{G(X)}{G(\mathbf{1}-X)}\le\frac{1}{n^2}\left(\sum_{i=1}^{n}x_i\right)\left(\sum_{i=1}^{n}\frac{1}{x_i(1-x_i)}\right)-\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1-x_i}$$

and

$$\log\frac{A(X)}{A(\mathbf{1}-X)}-\log\frac{I(X)}{I(\mathbf{1}-X)}\ge\frac{1-2\bar{x}}{2n^2(n+1)\bar{x}^2(1-\bar{x})^2}\left(n\sum_{i=1}^{n}x_i^2-\left(\sum_{i=1}^{n}x_i\right)^2\right),$$

respectively.

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