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DYNAMICS ON $(P_{cp}(X), H_d)$ GENERATED BY A SET OF DYNAMICS ON (X, d)

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Abstract. In this paper we study the following problem: Let (X, d) be a complete metric space. Let $f_1, \ldots, f_m : X \to X$ be some continuous weakly Picard operators. These operators generates the following operator

$$T_f : P_{cp}(X) \to P_{cp}(X), \quad A \mapsto f_1(A) \cup \cdots \cup f_m(A).$$

Is the operator $T_f : (P_{cp}(X), H_d) \to (P_{cp}(X), H_d)$ weakly Picard operator?

1. **Introduction**

Let X be a nonempty set and $f_1, \ldots, f_m : X \to X$ some operators. These operators generate the following operator on $P(X)$

$$T_f : P(X) \to P(X), \quad T_f(A) := f_1(A) \cup \cdots \cup f_m(A).$$

The problem is to study the operator T_f depending on the properties of the operators $f_1, \ldots, f_m$. In what follow we shall study this problem from the point of view of the Picard operators theory.

Throughout this paper we follow terminologies and notations in [27] and [36]. See also [31], [32] and [34]. For the multivalued operator theory see [36], [2], [21], [23].

2. **Iterated Picard operator systems**

We begin our study with the following open problem

Problem 1. (see [32] and [34]) Let (X, d) be a complete metric space and $f_1, \ldots, f_m : X \to X$ continuous Picard operators. Is the operator $T_f : (P_{cp}(X), H_d) \to (P_{cp}(X), H_d)$ Picard operator?

For the Problem 1 we have the following partial results:

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Theorem 2.1. (see [1], [13], [6], [42]) *If the operators* $f_1, \ldots, f_m$ *are a-contraction, then the operator*

$$T_f : P_{cp}(X) \to P_{cp}(X)$$

is an a-contraction.

Remark 2.1. By definition, the unique fixed point of T_f is the attractor of the iterated operator systems (IOS) $f_1, \ldots, f_m$.

Theorem 2.2. (see [33]) *If the operators* $f_1, \ldots, f_m$ *are* φ*-contractions, then the operator* $T_f : P_{cp}(X) \to P_{cp}(X)$ *is a* φ*-contraction.*

Theorem 2.3. (see [24]) *If the operators* $f_1, \ldots, f_m$ *are of Meir-Keeler type, then the operator* $T_f : P_{cp}(X) \to P_{cp}(X)$ *is a Meir-Keeler type operator.*

The following open problems are in connection with the Problem 1.

Problem 2. Let X be a nonempty set and $f_1, \ldots, f_m$ Bessaga operators. Does there exist $Y \subset P(X)$ such that

(a) $T_f(Y) \subset Y$,

(b) $T_f : Y \to Y$ is Bessaga operator?

Problem 3. Let X be a nonempty set and $f_1, \ldots, f_m$ Janos operators. Does there exist $Y \subset P(X)$ such that

(a) $T_f(T) \subset Y$,

(b) $T_f : Y \to Y$ is Janos operator?

Problem 4. Let (X, d) be a complete metric space and $f_1, \ldots, f_m : X \to X$ continuous Bessaga operators. Is the operator $T_f : P_{cp}(X) \to P_{cp}(X)$ Bessaga operator?

Problem 5. Let (X, d) be a complete metric space and $f_1, \ldots, f_m : X \to X$ continuous Janos operators. Is the operator $T_f : P_{cp}(X) \to P_{cp}(X)$ Janos operator?

In the case $m = 1$ we have

Example 2.1. Let $f : R \to R$, $f(x) = \frac{1}{2}x$ and $T_f : P(R) \to P(R)$, $T_f(A) = f(A)$. We remark that f is Bessaga operator (f is $\frac{1}{2}$-contraction), but $cardF_{T_f} > 1$. For example $\{0\}, R, R_+, R_-, R_+^*, R_-^*, \{2^k | k \in Z\}$, are fixed points of T_f.

Theorem 2.4. *Let* X *be a nonempty set and* $f : X \to X$ *a Bessaga operator. Then there exists* $Y \subset P(X)$ *such that* *(a)* $T_f(Y) \subset Y$

(b) $T_f : Y \to Y$ *is Bessaga operator.*

If $cardX > 1$, *then there exists* $Y \subset P(X)$ *such that* $cardY > 1$.

Proof. Here T_f is the following operator, $T_f : P(X) \to P(X)$, $T_f(A) = f(A)$. By a theorem of Bessaga ([27]) there exists a metric d on X such that (X, d) is a complete metric space and $f : (X, d) \to (X, d)$ is an a-contraction. By a theorem of Nadler ([22]) the operator $T_f : (P_{cp}(X), H_d) \to (P_{cp}(X), H_d)$ is an a-contraction. By the contraction principle $T_f|_{P_{cp}(X)}$ is Picard operator. So, $T_f|_{P_{cp}(X)}$ is Bessaga operator ($Y = P_{cp}(X, d)$).

Theorem 2.5. *Let* (X, d) *be a compact metric space and* $f : X \to X$ *continuous Janos operator. Then the operator* $T_f : P_{cp}(X) \to P_{cp}(X)$ *is Janos operator.*

Proof. By a theorem of Janos ([27]) there exists an equivalent metric (with d) ρ on X such that $f : (X, \rho) \to (X, \rho)$ is an a-contraction. By a theorem of Nadler ([22]) the operator $T_f : (P_{cp}(X), H_\rho) \to (P_{cp}(X), H_\rho)$ is an a-contraction, These imply that

$$\delta_{H_\rho} : (T_f(P_{cp}(X))) \leq a\delta_{H_\rho}(P_{cp}(X))$$

and

$$\delta_{H_\rho} : (T_f^n(P_{cp}(X))) \leq a^n\delta_{H_\rho}(P_{cp}(X)).$$

So

$$\bigcap_{n \in N} T_f^n(P_{cp}(X)) = \{\{x^*\}\}$$

where x^* is the unique fixed point of f.

3. Iterated weakly Picard operator systems

The basic problem of this paper is the following

Problem 6. Let (X, d) be a complete metric space and $f_1, \ldots, f_m : X \to X$ continuous WPOs. Is the operator $T_f : P_{cp}(X) \to P_{cp}(X)$ WPO?

The following open problems are in connection with the Problem 6.

Problem 7. Let (X, d) be a complete metric space and $f_1, \ldots, f_m \in C(X, X)$. We suppose that

$$F_{f_i} = F_{f_i^n} \neq \emptyset, \quad i = \overline{1, m},\ n \in N^*.$$

We ask if

$$F_{T_f} = F_{T_f^n} \neq \emptyset, \quad n \in N^*.$$

Problem 8. Let (X, d) be a compact metric space and $f_1, \ldots, f_m \in C(X, X)$. We suppose that

$$\bigcap_{n \in N} f_i^n(X) = F_{f_i}, \quad i = \overline{1, m}.$$

Does the operator T_f satisfy the condition

$$\bigcap_{n \in N} T_f^n(P_{cp}(X)) = F_{T_f}?$$

Problem 9. (see [4], [26]) Let (X, d) be a complete metric space and $f_i \in C(X, X)$, $i = \overline{1, m}$. We suppose that

$$\omega_{f_i}(x) \neq \emptyset, \ \forall \ x \in X, \ \forall \ i = \overline{1, m}.$$

Does this imply that

$$\omega_{T_f}(A) \neq \emptyset, \ \forall \ A \in P_{cp}(X)?$$

Problem 10. (see [4], [26]) Let (X, d) be a complete metric space and $f_i \in C(X, X)$, $i = \overline{1, m}$. If there exists $x \in X$ such that the recurrent point set of f_i,

$$R_{f_i}^{(x)} \neq \emptyset, \quad i = \overline{1, m},$$

does exist $A \in P_{cp}(X)$ such that

$$R_{T_f}(A) \neq \emptyset?$$

In the case $m = 1$, we have

Example 3.1. Let X be a Banach space, $K \in C([a, b] \times [a, b] \times X, X)$, $K(t, s, \cdot) : X \to X$ a L_K-Lipschitz operator, for all $t, s \in [a, b]$. Let $f : C([a, b], X) \to C([a, b], X)$ be defined by

$$f(x)(t) = x(a) + \int_a^t K(t, s, x(s))ds.$$

Let $X_\alpha := \{x \in C([a, b], X) | x(a) = \alpha\}, \ \alpha \in X$. Then

- $X = \bigcup X_\alpha$ is a partition of X,
- f is continuous,
- $X_\alpha \in I_{cl}(f)$,

- $f|_{X_\alpha}$ is a Picard operator, $\alpha \in X$,
- $T_f : P_{cp}(X_\alpha) \to P_{cp}(X_\alpha)$ is Picard operator, $\alpha \in X$,
- $T_f : \bigcup_{\alpha \in X} P_{cp}(X_\alpha) \to \bigcup_{\alpha \in X} P_{cp}(X_\alpha)$ is WPO with respect to the generalized Hausdorff-Pompeiu metric.

More general we have

Theorem 3.1. *Let (X, d) be a complete metric space, $X = \bigcup_{\alpha \in J} X_\alpha$ a partition of X, $f : X \to X$ a continuous operator such that:*

(i) $X_\alpha \in I_{cl}(f)$,

(ii) $f : X_\alpha \to X_\alpha$ is a-contraction, for all $\alpha \in J$.

Then there exists $S(X) \subset P(X)$ such that:

(i) $S(X) \in I(T_f)$,

(ii) $T_f : S(X) \to S(X)$ is WPO with respect to the generalized Hausdorff-Pompeiu metric on $S(X)$.

Proof. By a theorem of Nadler $T_f : P_{cp}(X_\alpha) \to P_{cp}(X_\alpha)$ is a-contraction for all $\alpha \in J$. Let $S(X) := \bigcup_{\alpha \in J} P_{cp}(X_\alpha)$. Then for all $A \in S(X)$, $T_f^n(A)$ converges to $T_f^\infty(A)$. If $A \in P_{cp}(A_\alpha)$, then $T_f^\infty(A) \in P_{cp}(X_\alpha)$, and is the unique fixed point of T_f in $P_{cp}(X_\alpha)$.

4. **Attractor and sequences of contractions**

Let (X, d) be a complete metric space, $f_1, \dots, f_m : X \to X$ a-contractions. Then $T_f : P_{cp}(X) \to P_{cp}(X)$ is a-contraction. By definition the unique fixed point of T_f, A^*, is the attractor of the iterated operator system $f_1, \dots, f_m$. The attractor A^* has the following properties (see [13], [43], [1],...):

a)

(i) $\emptyset \neq A^*$ is compact,

(ii) $f_i(A^*) \subset A^*$, for $1 \leq i \leq m$,

(iii) A^* is minimal with respect to (i) and (ii).

b) for all $x \in A^*$, there exists a sequence $i_1, \dots, i_s, \dots$ such that

$$f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_s}(y) \to x \text{ as } s \to \infty,$$

for all $y \in X$.

The above properties of the attractor give rise to the following problems:

Problem 11. Let $(X < d)$ be a complete metric space and $f, f_n : X \to X$, $n \in N$. We suppose that

(i) f and f_n are a-contractions, $n \in N$,

(ii) $f_n \xrightarrow{d} f$.

Does f_n^∞ converges to f^∞?

Problem 12. Let (X, d) be a complete metric space and $f, f_n : X \to X$ WPOs, $n \in N$. If $(f_n)_{n\in N}$ converges to f, does $(f_n^\infty)_{n\in N}$ converges to f^∞?

Problem 13. Let (X, d) be a complete metric space and $f_1, \ldots, f_m : X \to X$ φ-contractions. Let $(g_n)_{n\in N}$ a sequence in $\{f_1, \ldots, f_m\}$. Does converge the sequences

$$x_n := (g_0 \circ \cdots \circ g_n)(x)$$

and

$$y_n := (g_n \circ \cdots \circ g_0)(x)?$$

Problem 14. Let (X, d) be a complete metric space and $f_n : X \to X$ a r_n-contraction, $n \in N$. If $r_n \to 0$ as $n \to \infty$, does f_n converges to a constant operator?

We have the following result for the above problems

Theorem 4.1. (see [28]) *Let (X, d) be a complete metric space and $f, f_n : X \to X$, $n \in N$. We suppose that:*

(a) f is Picard operator ($F_f = \{x^*\}$);

(b) the sequence $(f_n)_{n\in N}$ is asymptotical uniform convergent to f;

(c) $F_{f_n} \neq \emptyset$, for all $n \in N$.

If $x_n^* \in F_{f_n}$, then $x_n^* \to x^*$ as $n \to \infty$.

Proof. By definition the sequence $(f_n)_{n\in N}$ is asymptotical uniform convergent to f if for all $\varepsilon > 0$ there exist $n_0(\varepsilon), m_0(\varepsilon)$ such that

$$d(f_n^m(x), f^m(x)) < \varepsilon$$

for all $n \geq n_0(\varepsilon)$, $m \geq m_0(\varepsilon)$ and all $x \in X$.

We have

$$d(x_n^*, x^*) = d(f_n^m(x_n^*), f^m(x^*)) \leq$$

$$\leq d(f_n^m(x_n^*), f^m(x_n^*)) + d(f^m(x_n^*), f^m(x^*)).$$

Let $\varepsilon > 0$ and $n_0(\varepsilon), m_0(\varepsilon)$ such that

$$d(f_n^m(x_n^*), f^m(x_n^*)) \leq \frac{\varepsilon}{2},$$

for all $n \geq n_0(\varepsilon),\ m \geq m_0(\varepsilon)$.

On the other hand for each $n \geq n_0(\varepsilon)$ there exists $m_n(\varepsilon)$ such that

$$d(f^{m_n(\varepsilon)}(x_n^*), x^*) < \frac{\varepsilon}{2}.$$

Theorem 4.2. (see [1], [5], [18]) *Let (X, d) be a complete metric space and $f_n : X \to X$ a α_n-contraction, such that $\alpha_n \to 0$ as $n \to \infty$. Let $x^* \in X$. Then the following statements are equivalent:*

(i) there exists $x_0 \in X$ such that $f_n(x_0) \to x^*$ as $n \to \infty$;

(ii) $f_n(x) \to x^*$ as $n \to \infty$, for all $x \in X$;

(iii) $x_n^* \to x^*$ as $n \to \infty$, where x_n^* is the unique fixed point of f_n.

Proof. (i) $\Rightarrow$ (ii). From the condition (i) we have

$$d(f_n(x), x^*) \leq d(f_n(x), f_n(x_0)) + d(f_n(x_0), x^*) \leq$$

$$\leq a_n d(x, x_0) + d(f_n(x_0), x^*).$$

(ii) $\Rightarrow$ (ii). We have

$$d(x_n^*, x^*) \leq d(f_n(x_n^*), f_n(x^*)) + d(f_n(x^*), x^*) \leq$$

$$\leq \alpha_n d(x_n^*, x^*) + d(f_n(x^*), x^*).$$

So

$$d(x_n^*, x^*) \leq \frac{1}{1 - \alpha_n} d(f_n(x^*), x^*) \to 0 \text{ as } n \to \infty.$$

(iii) $\Rightarrow$ (i). It follows from

$$d(f_n(x^*), x^*) \leq d(f_n(x^*), f_n(x_n^*)) + d(f_n(x_n^*), x^*) \leq$$

$$\leq (\alpha_n + 1) d(x_n^*, x^*).$$

Remark 4.1. For other results for the Problem 11-14 see [1], [5], [10], [22], [18], [19], [28], [36].

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