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SEMILINEAR EQUATIONS IN HILBERT SPACES WITH QUASI-POSITIVE NONLINEARITY

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Abstract. The problem is to show that $Ax + F(x) = 0$ has a solution, where A is linear, maximal monotone and the nonlinearity F is a quasi-positive operator of Leray-Schauder type. The existence result is obtained as a consequence of the properties of the Leray-Schauder degree. Finally, some applications are given.

1. **Introduction**

Let H be a real Hilbert space with the inner product denoted by $< \cdot, \cdot >$ and the corresponding norm

$$\|x\| = \sqrt{< x, x >} \quad , \qquad x \in H.$$

Let us consider the semilinear equation

$$Ax + F(x) = 0, \tag{1.1}$$

where $A : D(A) \subset H \to H$ is a densely defined linear operator and $N : H \to H$ is nonlinear. We establish an existence and uniqueness result for the equation (1.1) under some monotonicity conditions. Moreover, assume that A is maximal monotone. Equations of the form (1.1) arise in natural way in the theory of elliptic equations or integro-differential equations.

An operator $F : H \to H$ is called quasi-positive if there exists $\alpha \in \mathbf{R}$ such that

$$< F(x), x > \geq \alpha \left\| F(x) \right\|^2 \quad , \quad \forall \, x \in H, \; x \neq 0. \tag{1.2}$$

This notion is close related with the angle-bounded operators. First, the angle-boundedness concept is defined for linear operators acting from a Banach space into its dual, then the definition can be extended to nonlinear operators. For details, see [7].

2. **The Results**

We give the following:

Lemma 2.1. *If* $F : H \to H$ *is a quasi-positive operator with* $\alpha > 1/2$, *then*

$$\|x - F(x)\| \leq \|x\| \quad , \quad \forall\, x \in H,\ x \neq 0.$$

Proof. We have:

$$\|x - F(x)\|^2 =< x - F(x), x - F(x) >=$$

$$= \|x\|^2 - 2 < F(x), x > + \|F(x)\|^2 \leq$$

$$\leq \|x\|^2 - (2\alpha - 1)\, \|F(x)\|^2 \leq \|x\|^2 . \quad \square$$

If A is linear, maximal monotone, then for all $\lambda > 0$, the operator $I + \lambda A$ is invertible with continuous inverse $(I + \lambda A)^{-1} : H \to H$ and

$$\left\|(I + \lambda A)^{-1}\right\| \leq 1.$$

For proof and further properties, see [3].

Now, the equation (1) can be written as

$$(I + A)x = x - F(x) \Leftrightarrow x = (I + A)^{-1}(x - F(x)),$$

or

$$x = T(x) \Leftrightarrow (I - T)(x) = 0, \tag{2.1}$$

where $T = (I + A)^{-1}(I - F)$.

If F is an operator of Leray-Schauder type, then $I - F$ is compact and consequently, T is compact, as the product of a continuous operator with a compact one.

Indeed, if $D \subset H$ is bounded and $(x_n)_{n\geq 1} \subset D$, then there exists x such that $(I - F)(x_{k_n}) \to (I - F)(x)$, at least on a subsequence. Further, $(I + A)^{-1}$ is continuous, so $Tx_{k_n} \to Tx$.

In conclusion, the operator $I - T$ is compact perturbation of the identity map and consequently, the Leray-Schauder degree can be considered.

Roughly speaking, the degree of ϕ at y, relative to D, denoted $d(\phi, D, y)$, is a measure of the number of the solutions of the equation $\phi(x) = y$ in D.

In an infinite dimensional Banach space X, the Leray-Schauder degree is defined for compact perturbations of the identity map, also named Leray-Schauder operators, $\phi \in (LS)$. Some properties of the Leray-Schauder degree are of interest in our work.

Proposition 2.1. *Let $\phi : D \subset X \to X$ be such that $I - \phi$ is compact and let $y \in X \setminus \phi(\partial D)$. Then the Leray-Schauder degree $d(\phi, D, y)$ satisfies the following properties:*

(a) *If $d(\phi, D, y) \neq 0$, then $y \in \phi(D)$.*

(b) *If $H \in C([0,1] \times D, X)$ is such that $I - H(t, \cdot)$ is compact, for all $t \in [0,1]$ and $y \in X \setminus H([0,1] \times \partial D)$, then the degree*

$$d(H(t,\cdot), D, y) = constant \quad , \quad \forall\, t \in [0,1].$$

(c) *The degree for the identity map $I : X \to X$ is*

$$d(I, D, y) = \begin{cases} 1 \quad , \quad y \in D \\ 0 \quad , \quad y \notin D \end{cases} .$$

For more details, see [4], [5].

Now, we can establish the following existence result:

Theorem 2.1. *Let $A : D(A) \subset H \to H$, $0 \in IntD(A)$, linear, maximal monotone and $F : H \to H$ be an (LS) - operator such that*

$$< F(x), x > \geq \alpha \left\| F(x) \right\|^2 \quad , \quad \forall\, x \in H,\ x \neq 0,$$

for some $\alpha > 1/2$. Then the equation $Ax + F(x) = 0$ has at least one solution $x \in D(A)$.

Proof. Let $B = B(0, r)$ be such that $\overline{B} \subset D(A)$. We have seen that the equation $Ax + F(x) = 0$ is equivalent with

$$(I - T)(x) = 0,$$

where $T = (I + A)^{-1}(I - F)$ is compact.

Let us consider the Leray-Schauder homotopy

$$H(t, x) = x - tT(x) \quad , \quad x \in \overline{B},\ t \in [0,1].$$

If $0 \in H(1, \partial B)$, the conclusion follows immediately. In order to use the invariance to homotopy of the Leray-Schauder degree, we prove that $0 \notin H([0,1), \partial B)$. Let us suppose by contrary that $H(t,x) = 0$, for some $x \in \partial B$ and $t \in [0,1)$. It results

$$\|x\| = t\,\|T(x)\| \leq \|T(x)\| = \left\|(I+A)^{-1}(I-F)\right\| \leq$$

$$\leq \left\|(I+A)^{-1}\right\| \cdot \|x - F(x)\| \leq \|x - F(x)\| \leq \|x\|.$$

We must have equalities all over, in particular $T(x) = 0$. Hence $x = 0 \in \partial B$, contradiction. This means that $0 \notin H([0,1], \partial B)$ and further,

$$d(H(1,\cdot), B, 0) = d(H(0,\cdot), B, 0) \Rightarrow$$

$$\Rightarrow d(I - T, B, 0) = d(I, B, 0) = 1.$$

In conclusion, $d(I - T, B, 0) \neq 0$, thus the equation $(I - T)(x) = 0$ and equivalent, the equation $Ax + F(x) = 0$ has at least one solution in $D(A)$. $\square$

3. **An Application**

Now, we are in position to show how the theoretical results from the previous section can be applied to the elliptic boundary value problems.

Let $\Omega \subset \mathbf{R}^n$ be open, bounded and let $a_{ij} \in C^1(\overline{\Omega})$, $1 \leq i,j \leq n$ be real valued functions satisfying the ellipticity property

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq 0 \quad , \quad \forall\, \xi = (\xi_1, ..., \xi_n) \in \mathbf{R}^n.$$

Let us consider the following elliptic problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(t)\frac{\partial x}{\partial x_i}\right) + g(t,x) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

The particular case $g(t,x) = a_0(t)x$, with $a_0 \in C(\overline{\Omega})$, $a_0 > p > 0$, is studied in [3], using Lax-Milgram theorem. Some existence results are also obtained in [1] and [2], as a consequence of some general considerations about saddle points. The general case of problem (3.1) is studied in [6], under the assumption that the nonlinear part is strongly monotone.

Here we assume that g satisfies

$$\int_\Omega g(t,x(t))\cdot x(t)\,dt \geq \alpha \int_\Omega g^2(t,x(t))\,dt, \tag{3.2}$$

for some $\alpha > 1/2$. Remark that in case $g(t,x) = a_0(t)x$, the condition (3.2) is fulled with $\alpha < 1/\left\| a_0 \right\|$.

Under the condition (3.2), the problem (3.1) has at least one solution in weak sense. Indeed, we can apply theorem 2.1 in the following functional background:

$$H = L^2(\Omega)\ ,\ Ax = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(t)\frac{\partial x}{\partial x_i}\right)\quad ,\quad D(A) = H^2(\Omega)\cap H_0^1(\Omega)$$

and $(Fx)(t) = g(t,x)$. The problem (3.1) can be written in the abstract form

$$Ax + F(x) = 0\ \ ,\quad x \in D(A) \subset L^2(\Omega).$$

We have:

$$< Ax, x >= \int_\Omega \sum_{i,j=1}^{n} a_{ij}\frac{\partial x}{\partial x_j}\cdot\frac{\partial x}{\partial x_i} \geq 0,$$

and $I + A$ is surjective, e.g. [2], therefore A is maximal monotone.

Finally, if g is compact perturbation of the identity, then the assertion is proved.

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