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ON GAUSS TYPE FUNCTIONAL EQUATIONS AND MEAN VALUES BY H. HARUKI AND TH. M. RASSIAS

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Abstract. In this paper we give a concise summary of some recent results on Gauss type functional equations and mean values by H. Haruki and Th. M. Rassias.

1. Introduction

Ten years ago, in [5] Haruki reconsidered the Gauss' functional equation

$$f\left(\frac{a+b}{2}, \sqrt{ab}\right) = f(a,b) \quad (a, b > 0), \tag{1.1}$$

where $f : R^+ \times R^+ \to R$ is an unknown function.

It is well known that $f(a,b) = AG(a,b)$ satisfies (1.1) where $AG(a,b)$ is the airhtmetic-geometric mean of Gauss of a, b defined as the common limit of the sequences (a_n), (b_n) given recurrently by

$$a_0 = a, \quad b_0 = b, \quad a_{n+1} = (a_n + b_n)/2, \quad b_{n+1} = \sqrt{a_n b_n}.$$

The result given by Haruki may be stated as follows.

Theorem 1.1. *Let $f : R^+ \times R^+ \to R$. If f can be represented by the form, containing some function p, in $R^+ \times R^+$*

$$f(a,b) = \frac{1}{2\pi}\int_0^{2\pi} p(r)d\theta,$$

where $p : R^+ \to R$ and $p''(x)$ is continuous in R^+, then the only solution of (1.1) is given by

$$f(a,b) = c_1 \frac{1}{AG(a,b)} + c_2, \tag{1.2}$$

where c_1 and c_2 are arbitrary real numbers.

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It should be noted that Gauss established an integral representation of $AG(a,b)$ as

$$AG(a,b) = \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{\sqrt{a^2 \cos^2\theta + b^2 \sin^2\theta}} \right)^{-1}. \tag{1.3}$$

So, (1.2) can be represented by using (1.3) as

$$f(a,b) = \frac{c_1}{2\pi} \int_0^{2\pi} \frac{d\theta}{\sqrt{a^2 \cos^2\theta + b^2 \sin^2\theta}} + c_2.$$

May be motivated by this fact, in [5] Haruki considered the following type mean value of a, b

$$M(a,b;p(r)) := p^{-1} \left(\frac{1}{2\pi} \int_0^{2\pi} p(r) d\theta \right),$$

where $p : R^+ \to R$, $p''(x)$ is a continuous function in R^+, $p = p(x)$ is strictly monotonic in R^+, and denote $\sqrt{a^2 \cos^2\theta + b^2 \sin^2\theta}$ by r.

The following theorem was proved in [5].

Theorem 1.2. *Let $c_1 (\neq 0)$ and c_2 be arbitrary real constants.*

(i) $M(a,b;p(r)) = AG(a,b)$ holds for all positive real numbers a, b if and only if $p(r) = c_1(1/r) + c_2$.

(ii) $M(a,b;p(r)) = G(a,b)$ holds for all positive real numbers a, b if and only if $p(r) = c_1(1/r^2) + c_2$.

(iii) $M(a,b;p(r)) = A(a,b)$ holds for all positive numbers a, b if and only if $p(r) = c_1 \log r + c_2$.

(iv) $M(a,b;p(r)) = \sqrt{\frac{a^2+b^2}{2}}$ holds for all positive real numbers a, b if and only if $p(r) = c_1 r^2 + c_2$.

(v) There exists no $p(r)$ such that $M(a,b;p(r)) = H(a,b)$ holds for all positive real numbers a, b.

Since then, around the above two theorems, a series of new generalization appeared one after another.

We would like to make a survey in this paper.

Throughout this paper, let a and b be two any positive real numbers. A mean value of a, b, denoted by $M(a,b)$ is defined to be a real-valued function M, which satisfies the following postulates:

(P_1) $M : R^+ \times R^+ \to R$;

(P_2) $M(a,b) = M(b,a)$ (symmetry property);

(P_3) $M(a, a) = a$ (reflexivity property).

The arithmetic, geometric, and harmonic mean values of a, b are denoted by $A(a, b)$, $G(A, b)$ and $H(a, b)$, respectively.

In what follows, we also use the power means defined by

$$P_q(a, b) = \left(\frac{a^q + b^q}{2}\right)^{\frac{1}{q}}$$

for $q \neq 0$, while, for $q = 0$,

$$P_0(a, b) = G(a, b).$$

We denote also the power function

$$e_n(x) = x^n \text{ for } n \neq 0$$

and

$$e_0(x) = \log x.$$

2. **Gauss Type Functional Equations**

$$f\left(\frac{a+b}{2}, \frac{2ab}{a+b}\right) = f(a, b) \quad (a, b > 0), \tag{2.1}$$

where $f : R^+ \times R^+ \to R$ is an unknown function of the above equation. By following the theory on Gauss' functional equation (cf. [1], [2], [3], [4]), a new result on this functional equation is given as

Theorem 2.1. *Let $f : R^+ \times R^+ \to R$ be a function. If f can be represented by*

$$f(a, b) = \frac{1}{2\pi}\int_0^{2\pi} q(s)d\theta \quad (a, b > 0),$$

where $s = a\cos^2\theta + b\sin^2\theta$, $q : R^+ \to R$ is a function such that $q''(x)$ is continuous in R^+, then the only solution of (2.1) is given by

$$f(a, b) = c_1\frac{1}{\sqrt{ab}} + c_2,$$

where c_1 and c_2 are arbitrary real numbers.

An open problem for the functional equation (2.1) is given as follows:

Let $f : R^+ \times R^+ \to R$ be a continuous function in $R^+ \times R^+$. Is the only continuous solution of the functional equation (2.1) given by

$$f(a, b) = F(ab),$$

where $F : R^+ \to R$ is an arbitrary continuous function of a real variable x?

In [13], the author treat the functional equation

$$f\left(\sqrt{ab}, \frac{2ab}{a+b}\right) = f(a,b) \quad (a,b>0), \tag{2.2}$$

where $f : R^+ \times R^+ \to R$ is an unknown function of the above equation.

By following the theory on Gauss' functional equation, we obtained

Theorem 2.2. *Let $f : R^+ \times R^+ \to R$ be a function. If f can be represented by*

$$f(a,b) = \frac{1}{2\pi}\int_0^{2\pi} u(t)d\theta \quad (a,b>0),$$

where $t = \left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)^{-\frac{1}{2}}$, $u : R^+ \to R$ is a function such that $u''(x)$ is continuous in R^+, then the only solution of (2.2) is given by

$$f(a,b) = c_1 GH(a,b) + c_2,$$

where c_1 and c_2 are arbitrary real numbers.

$GH(a,b)$ is the geometric-harmonic mean of a and b defined as the common limit of the sequences (a_n), (b_n) given recurrently by

$$a_0 = a, \quad b_0 = b, \quad a_{n+1} = \sqrt{a_n b_n}, \quad b_{n+1} = \frac{2a_n b_n}{a_n + b_n}.$$

Also, an open problem for the functional equation (2.2) is given as follows:

Let $f : R^+ \times R^+ \to R$ be a continuous function in $R^+ \times R^+$. Is the only continuous solution of the functional equation (2.2) given by

$$f(a,b) = F(GH(a,b)),$$

where $F : R^+ \to R$ is an arbitrary continuous function of a real variable x?

In [16], G. Toader considered a more general functional equation

$$f(P_q(a,b), P_s(a,b)) = f(a,b). \tag{2.3}$$

Denote

$$r_n(\theta) = (a^n\cos^2\theta + b^n\sin^2\theta)^{1/n}, \quad n \neq 0$$

and

$$r_0(\theta) = \lim_{n\to 0} r_n(\theta) = a^{\cos^2\theta} b^{\sin^2\theta}.$$

For a strictly monotonic function $p : R^+ \to R$, consider the function

$$f(a,b;p,n) = \frac{1}{2\pi}\int_0^{2\pi} p(r_n(\theta))d\theta. \tag{2.4}$$

G. Toader proved the following theorem.

Theorem 2.3. *If the function f is a solution of (2.3) which can be represented by (2.4), where p has a continuous second-order derivative in R^+, then*

$$p = c_1 e_{q+s-n} + c_2, \tag{2.5}$$

where c_1 and c_2 are arbitrary real numbers.

Remark. For $n = 2$, $q = 1$ and $s = 0$, we get the necessity part of Theorem 1.1. For $n = 1$, $q = 1$ and $s = -1$, we get the necessity part of Theorem 2.1. For $n = -2$, $q = 0$ and $s = -1$, we get the necessity part of Theorem 2.2. In all these three cases, as we have already mentioned, the condition is also sufficient.

In [17], the following theorem was proved.

Theorem 2.4. *If $n \neq 0$, $q = n$ and $s = -n$, then the function f given by (2.4) and p given by (2.5), verifies the relation (2.3).*

In [10], Kim and Rassias considered a generalized functional equation, namely

$$f(P_q^k(a,b), P_s^k(a,b)) = f(a,b) \tag{2.6}$$

where

$$P_q^k(a,b) = (ab)^{(1-k)/2}\left(\frac{a^q+b^q}{2}\right)^{\frac{k}{q}}.$$

The following theorem was proved.

Theorem 2.5. *If the function f is a solution of (2.6) which can be represented by (2.4), where p has a continuous second-order derivative in R^+, then*

$$p = c_1 e_{-n+kq+ks} + c_2,$$

where c_1 and c_2 are arbitrary real numbers.

Clearly, Theorem 2.3 is a special case of Theorem 2.5.

In [18], S. Toader, Rassias and G. Toader consider a more general functional equation

$$f(M(a,b), N(a,b)) = f(a,b), \tag{2.7}$$

where M and N are two given means.

It is not difficult to prove the following theorem.

Theorem 2.6. *If the function f defined by (2.4) in case $n = 1$ is a solution of (2.6), where p has a continuous second-order derivative in R^+, then the function p is a solution of the differential equation*

$$p''(c) + 4p'(x)[M''_{ab}(c,c) + N''_{ab}(c,c)] = 0.$$

Remark. In case $n = 1$, Theorem 2.3 and Theorem 2.5 can be deduced from Theorem 2.6.

3. Mean Values by H. Haruki and Th.M. Rassias

In [7], Haruki and Rassias considered the following two mean values of a, b:

$$M(a,b;q(s)) := q^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} q(s)d\theta\right),$$

where $q : R^+ \to R$, $q''(x)$ is a continuous function in R^+, $q = q(x)$ is strictly monotonic in R^+, and denote $a\cos^2\theta + b\sin^2\theta$ by s; and

$$M(a,b;u(t)) := u^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} u(t)d\theta\right),$$

where $u : R^+ \to R$, $u''(x)$ is a continuous function in R^+, $u = u(x)$ is strictly monotonic in R^+, and denote $(\cos^2\theta/a + \sin^\theta/b)^{-1}$ by t.

The following two theorems are proved.

Theorem 3.1. *Let $c_1(\neq 0)$ and c_2 be arbitrary real constants.*

(i) $M(a,b;q(s)) = A(a,b)$ holds for all positive real numbers a, b if and only if $q(s) = c_1 s + c_2$.

(ii) $M(a,b;q(s)) = G(a,b)$ holds for all positive real numbers a, b if and only if $q(s) = c_1(1/s) + c_2$.

(iii) $M(a,b;q(s)) = P_{\frac{1}{2}}(a,b)$ holds for all positive real numbers a, b if and only if $q(s) = c_1 \log s + c_2$.

(iv) $M(a,b;q(s)) = \sqrt{H(a,b)G(a,b)}$ holds for all positive real numbers a, b if and only if $q(s) = c_1(1/s^2) + c_2$.

Theorem 3.2. *Let $c_1(\neq 0)$ and c_2 be arbitrary real constants.*

(i) $M(a,b,u(t)) = G(a,b)$ holds for all positive real numbers a, b if and only if $u(t) = c_1 t + c_2$.

(ii) $M(a,b,u(t)) = H(a,b)$ *holds for all positive real numbers* a,b *if and only if* $u(t) = c_1(1/t) + c_2$.

(iii) $M(a,b,u(t)) = P_{-\frac{1}{2}}(a,b)$ *holds for all positive real numbers* a,b *if and only if* $u(t) = c_1 \log s + c_2$.

(iv) $M(a,b,u(t)) = \sqrt{A(a,b)G(a,b)}$ *holds for all positive real numbers* a,b *if and only if* $u(t) = c_1 t^2 + c_2$.

Noticed that the geometric-harmonic mean $GH(a,b)$ can be represented by a first complete elliptic integral as

$$GH(a,b) = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{\sqrt{\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}}}, \tag{3.1}$$

the author in [12] considered the mean value of a,b

$$M(a,b;v(z)) = v^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} v(z)d\theta\right),$$

where $v : R^+ \to R$, $v''(x)$ is a continuous function in R^+, $v = v(x)$ is strictly monotonic in R^+, and denote $(\cos^2\theta/a^2 + \sin^2\theta/b^2)^{-\frac{1}{2}}$ by z.

The following theorem is proved.

Theorem 3.3. *Let* $c_1(\neq 0)$ *and* c_2 *be arbitrary real constants.*

(i) $M(a,b;v(z)) = GH(a,b)$ *holds for all positive real numbers* a,b *if and only if* $v(z) = c_1 z + c_2$.

(ii) $M(a,b;v(z)) = G(a,b)$ *holds for all positive real numbers* a,b *if and only if* $v(z) = c_1 z^2 + c_2$.

(iii) $M(a,b;v(z)) = H(a,b)$ *holds for all positive real numbers* a,b *if and only if* $v(z) = c_1 \log z + c_2$.

(iv) $M(a,b;v(z)) = (H(a^2,b^2))^{1/2}$ *holds for all positive real numbers* a,b *if and only if* $v(z) = c_1(1/z^2) + c_2$.

(v) There exists no $v(z)$ *such that* $M(a,b;v(z)) = A(a,b)$ *holds for all positive real numbers* a,b.

It should be noted that in [8] Kim also considered the mean value $M(a,b;v(z))$ and got the results (ii), (iii), (iv) of Theorem 3.3.

In [16] and [17], G. Toader and Rassias considered a generalization of the above mentioned four mean values $M(a,b;p(r))$, $M(a,b;q(s))$, $M(a,b;u(t))$ and $M(a,b;v(z))$ as follows:

Denote

$$r_n(\theta) = (a^n \cos^2\theta + b^n \sin^2\theta)^{1/n}, \quad n \neq 0,$$

and

$$r_0(\theta) = \lim_{n\to 0} r_n(\theta) = a^{\cos^2\theta} b^{\sin^2\theta}.$$

For a strictly monotonic function $p : R^+ \to R$, set

$$M(a,b;p,r_n) = p^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} p(r_n(\theta))d\theta\right).$$

It is easy to prove that $M(a,b;p,r_n)$ is a mean value.

As was stated in Theorem 1.2, Theorem 3.1, Theorem 3.2 and Theorem 3.3, the means $M(a,b;p,r_n)$ can represent some known means for special choice of p and n. In [10], the following theorem was proved.

Theorem 3.4. *If for some twice continuously differentiable function p the mean $M(a,b;p,r_n)$ reduces at the power mean $P_q(a,b)$, then*

$$p = c_1 e_{2q-n} + c_2,$$

where c_1 and c_2 are arbitrary real numbers.

In [17], the following theorem was proved.

Theorem 3.5. *The mean $M(a,b;p,r_n)$ reduces to the power mean $P_q(a,b)$ for arbitrary n if*

$$p = c_1 e_{2q-n} + c_2, \quad c_1, c_2 \in R$$

and q takes one of following values; (i) $q = 0$, (ii) $q = n$; or (iii) $q = n/2$.

In [9], Kim considered some further extensions of values by H. Haruki and Th.M. Rassias as follows:

$$M(a,b;h(s)) := \frac{1}{H(a,b)} h^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} h(s)d\theta\right), \tag{3.2}$$

where $h : R^+ \to R$, $h''(x)$ is a continuous function in R^+, $h = h(x)$ is strictly monotonic in R^+, and denote $(\cos^2\theta/a^2 + \sin^2\theta/b^2)^{-1}$ by s,

$$M(a,b;k(s)) := \frac{1}{H(a,b)} k^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} k(s)d\theta\right), \tag{3.3}$$

where $k : R^+ \to R$, $k''(x)$ is a continuous function in R^+, $k = k(x)$ is strictly monotonic in R^+, and denote $(a\cos\theta)^2 + (b\sin\theta)^2$ by s.

The following theorems are proved:

Theorem 3.6. *Let $c_1(\neq 0)$ and c_2 be arbitrary real constants.*

(i) $M(a, b; h(s)) = A(a, b)$ holds for all positive real numbers a, b if and only if $h(s) = c_1 s + c_2$.

(ii) $M(a, b; h(s)) = ab(a+b)/(a^2 + b^2)$ holds for all positive real numbers a, b if and only if $h(s) = c_1(1/s) + c_2$.

(iii) $M(a, b; h(s)) = H(a, b)$ holds for all positive real numbers a, b if and only if $h(s) = c_1 \log s + c_2$.

(iv) $M(a, b; h(s)) = \sqrt{2(a+b)^2(ab)^2/(3a^4 + 3b^4 + 2(ab)^2)}$ holds for all positive real numbers a, b if and only if $h(s) = c_1(1/s^2) + c_2$.

(v) $M(a, b; h(s)) = \sqrt{(a^2 + b^2)(a+b)^2/8ab}$ holds for all positive real numbers a, b if and only if $h(s) = c_1 s^2 + c_2$.

Theorem 3.7. *Let $c_1(\neq 0)$ and c_2 be arbitrary real constants.*

(i) $M(a, b; k(s)) = (a^2 + b^2)(a+b)/4ab$ holds for all positive real numbers a, b if and only if $k(s) = c_1 s + c_2$.

(ii) $M(a, b; k(s)) = A(a, b)$ holds for all positive real numbers a, b if and only if $k(s) = c_1(1/s) + c_2$.

(iii) $M(a, b; k(s)) = (a+b)^3/8ab$ holds for all positive real numbers a, b if and only if $k(s) = c_1 \log s + c_2$.

(iv) $M(a, b; k(s)) = \sqrt{(ab)(a+b)^2/2(a^2 + b^2)}$ holds for all positive real numbers a, b if and only if $k(s) = c_1(1/s^2) + c_2$.

(v) $M(a, b; k(s)) = \sqrt{(a+b)^2(3a^4 + 3b^4 + 2(ab)^2)/32(ab)^2}$ holds for all positive real numbers a, b if and only if $k(s) = c_1 s^2 + c_2$.

Instead of (3.2) and (3.3), in [14] the author considered in general, the following two mean values of a, b:

$$M(a, b; h(s), q) := \frac{1}{P_q(a, b)} h^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} h(s)d\theta\right), \tag{3.4}$$

and

$$M(a, b; k(s), q) := \frac{1}{P_q(a, b)} k^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} k(s)d\theta\right), \tag{3.5}$$

where $h(s)$ and $k(s)$ are just the same as in (3.2) and (3.3).

Moreover, denote

$$s_n(\theta) = (a^{2n}\cos^2\theta + b^{2n}\sin^2\theta)^{\frac{1}{n}}, \quad n \neq 0,$$

and

$$s_0(\theta) = \lim_{n\to 0} s_n(\theta) = a^{2\cos^2\theta} b^{2\sin^2\theta}.$$

If $p : R^+ \to R$ is a strictly monotonic function, then

$$M(a,b;p,s_n;q) = \frac{1}{P_q(a,b)} p^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} p(s_n(\theta))d\theta\right)$$

defines a mean value of a, b. Clearly, (3.4) is given for $n = -1$ and (3.5) is given for $n = 1$.

We have the following two theorems.

Theorem 3.8. *If for some twice continuously differentiable function p the mean $M(a,b;p,s_n;q)$ reduces at the power mean $P_r(a,b)$, then*

$$p = c_1 e_{(q+r)/2-n} + c_2,$$

where c_1 and c_2 are arbitrary real numbers.

Theorem 3.9. *The mean $M(a,b;p,s_n;q)$ reduces to the power mean $P_r(a,b)$ for arbitrary n if*

$$p = c_1 e_{(q+r)/2-n} + c_2, \quad c_1, c_2 \in R$$

and r takes one of the following values: (i) $r = -q$ or (ii) $r = q = n$.

In [10], Kim and Rassias considered a new mean value

$$M(a,b;p,r_{n,k}) := (ab)^{(1-k)/2} p^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} p(r_{n,k}(\theta))d\theta\right) \tag{3.6}$$

where $p : R^+ \to R$ is a strictly monotonic function, n and k are real numbers,

$$r_{n,k}(\theta) = (a^{kn}\cos^2\theta + b^{kn}\sin^2\theta)^{\frac{1}{n}}, \quad n, k \neq 0,$$

and

$$r_{0,k}(\theta) = \lim_{n\to 0} r_{n,k}(\theta) = a^{k\cos^2\theta} b^{k\sin^2\theta}, \quad k \neq 0.$$

The mean can represent some known means for special choice of p, k and n. Two well-known examples are given for $n = 2$, $k = 1$, $p(x) = x^{-1}$ and $n = -2$, $k = 1$, $p(x) = x$ respectively. They correspond to the arithmetic-geometric mean of Gauss (1.3) and geometric-harmonic mean (3.1) respectively.

Kim and Rassias in [10] also considered the following generalization of the power means defined by

$$H_q^k(a,b) = (ab)^{(1-k)/2}\left(\frac{2a^q b^q}{a^q+b^q}\right)^{k/q}, \quad k \neq 0$$

for $q \neq 0$, while $H_0^k(a,b) = \lim_{q\to 0} H_q^k(a,b) = \sqrt{ab}$ for $q = 0$.

It is not difficult to prove the following theorems.

Theorem 3.10. *If the mean $M(a,b;p,r_{n,k})$ reduces to the power mean $P_q^k(a,b) = H_{-q}^k(a,b)$ for some twice continuously differentiable function p, then*

$$p = c_1 e_{(2kq-nk^2)/k^2} + c_2,$$

where c_1 and c_2 are arbitrary real numbers.

Theorem 3.11. *The mean $M(a,b;p,r_{n,k})$ reduces to the power mean $P_q^k(a,b)$ for some arbitrary n if*

$$P = c_1 e_{(2kq-nk^2)/k^2} + c_2, \quad c_1, c_2 \in R$$

and q takes one of the following values: (i) $q = 0$, (ii) $q = nk$; or (iii) $q = nk/2$.

Theorem 3.12. *Let $c_1(\neq 0)$ and c_2 be arbitrary real constants.*

(i) $M(a,b;p,r_{1,k}) = \frac{1}{2}(a^k+b^k)(ab)^{(1-k)/2}$ holds for all positive real numbers a,b if and only if $p(s) = c_1 s + c_2$.

(ii) $M(a,b;p,r_{1,k}) = G(a,b)$ holds for all positive real numbers a,b if and only if $p(s) = c_1(1/s) + c_2$.

(iii) $M(a,b;p,r_{1,k}) = \frac{1}{4}(ab)^{(1-k)/2}(a^{k/2}+b^{k/2})^2$ holds for all positive real numbers a,b if and only if $p(s) = c_1 \log s + c_2$.

(iv) $M(a,b;p,r_{1,k}) = \dfrac{\sqrt{2}(ab)^{(k+2)/4}}{(a^k+b^k)^{1/2}}$ holds for all positive real numbers a,b if and only if $p(s) = c_1(1/s^2) + c_2$.

(v) $M(a,b;p,r_{1,k}) = \dfrac{[3(a^{2k}+b^{2k})+2(ab)^k]^{1/2}}{[8(ab)^{(k-1)}]^{1/2}}$ holds for all positive real numbers a,b if and only if $p(s) = c_1 s^2 + c_2$.

Theorem 3.13. *Let $c_1(\neq 0)$ and c_2 be arbitrary real constants.*

(i) $M(a,b;p,r_{-1,k}) = G(a,b)$ holds for all positive real numbers a,b if and only if $p(s) = c_1 s + c_2$.

(ii) $M(a,b;p,r_{-1,k}) = 2(ab)^{(k+1)/2}(a^k+b^k)^{-1}$ holds for all positive real numbers a,b if and only if $p(s) = c_1(1/s) + c_2$.

(iii) $M(a,b;p,r_{-1,k}) = 4(ab)^{(1+k)/2}(a^{k/2}+b^{k/2})^{-2}$ *holds for all positive real numbers* a,b *if and only if* $p(s) = c_1 \log s + c_2$.

(iv) $M(a,b;p,r_{-1,k}) = \dfrac{1}{\sqrt{2}}(a^k+b^k)^{1/2}(ab)^{(2-k)/4}$ *holds for all positive real numbers* a,b *if and only if* $p(s) = c_1(1/s^2) + c_2$.

(v) $M(a,b;p,r_{-1,k}) = \dfrac{[8(ab)^{k+1}]^{1/2}}{[3(a^{2k}+b^{2k})+2(ab)^k]^{1/2}}$ *holds for all positive real numbers* a,b *if and only if* $p(s) = c_1 s^2 + c_2$.

Instead of (3.6), Rassias and Kim in [15] introduce in general, the following mean values of a,b:

$$M(a,b;p,r_{n,k};q) := [P_q(a,b)]^{(1-k)}p^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} p(r_{n,k}(\theta))d\theta\right)$$

where $p(r_{n,k}(\theta))$ is just the same as in (3.6).

The following theorems are proved.

Theorem 3.14. *If the mean* $M(a,b;p,r_{n,k};q)$ *reduces to the power mean* $P_s(a,b)$ *for some twice continuously differentiable function* p*, then*

$$p = c_1 e_{\frac{2q(k-1)+2s}{k^2}-n} + c_2,$$

where c_1 *and* c_2 *are arbitrary real numbers.*

Theorem 3.15. *The mean* $M(a,b;p,r_{n,k};q)$ *reduces to the power mean* $P_s(a,b)$ *for some arbitrary* n *if*

$$p = c_1 e_{\frac{2q(k-1)+2s}{k^2}-n} + c_2, \quad c_1, c_2 \in R$$

and s *takes one of the following values: (i)* $s = q = 0$*, (ii)* $s = -q$, $k = 2$*, (iii)* $s = q = nk$*; or (iv)* $s = q = nk/2$.

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