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POINTWISE APPROXIMATION BY GENERALIZED SZÁSZ-MIRAKJAN OPERATORS

ZOLTÁN FINTA

Abstract. In this paper we establish some local approximation properties for a generalized Szász - Mirakjan - type operator.

1. **Introduction**

It is well - known the operator of Szász - Mirakjan [11] defined by

$$(S_n f)(x) \;=\; e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!}\, f\left(\frac{k}{n}\right), \tag{1}$$

where f is any function defined on $[0,\infty)$ such that $(S_n|f|)(x) < \infty$. The operator S_n was generalized by Pethe and Jain in [10], by Stancu in [12] and by Mastroianni in [7], obtaining S_n^α operators

$$(S_n^\alpha f)(x) \;=\; (1+n\alpha)^{-x/\alpha} \cdot \sum_{k=0}^{\infty} \left(\alpha + \frac{1}{n}\right)^{-k} \cdot \frac{x(x+\alpha)\ldots(x+(k-1)\alpha)}{k!}\, f\left(\frac{k}{n}\right), \tag{2}$$

where α is a nonnegative parameter depending on the natural number n and f is any real function defined on $[0,\infty)$ with $(S_n^\alpha|f|)(x) < \infty$. This operator has been also considered by Della Vecchia and Kocic' [3]. It was studied extensively the uniform convergency in compact interval, monotonicity, convexity, evaluation of the remainder in approximation formula and degeneracy property of the operators (2), respectively.

In [6] Lupaş has introduced the following operator:

$$(L_n f)(x) \;=\; 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k\, k!}\, f\left(\frac{k}{n}\right), \tag{3}$$

where $f : [0,\infty) \to R$, $(nx)_0 = 1$ and $(nx)_k = nx(nx+1)\ldots(nx+k-1),\ k \geq 1$. This operator was studied by Agratini [1] and Miheşan [8]. In fact we have $S_n^{1/n} = L_n$ [7, p. 250].

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The purpose of this paper is to establish pointwise approximation properties for the Szász - Mirakjan - type operator defined by (2).

In what follows we denote by $C_B[0,\infty)$ the set of all bounded and continuous functions on $[0,\infty)$ endowed with the norm $\|f\| = \sup\{\ |f(x)|\ :\ x \in [0,\infty)\ \}$. Let $\Delta_h^2(f,x) = f(x-h) - 2f(x) + f(x+h)$ $(x \ge h)$ be the usual symmetric second difference of f and $\omega^2(f,\delta) = \sup_{0<h\le\delta,\ x\ge h}\ |\Delta_h^2(f,x)|$ the modulus of smoothness of f.

2. **Main results**

The following results give some local approximation properties for S_n^α :

Theorem 1. *For every function $f \in C[0,\infty)$ we have*

$$|(S_n^\alpha f)(x) - f(x)| \ \le\ 2\ \omega^2\left(\ f\ ,\ \sqrt{(\alpha+\frac{1}{n})\frac{x}{2}}\ \right). \tag{4}$$

Proof. Let $e_0(x) = 1$ and $e_1(x) = x$ $(x \ge 0)$. In view of [7, p. 239, Theorem 2.3] we obtain that S_n^α reproduces every linear function and $(S_n^\alpha(e_1 - xe_0)^2)(x) = (\alpha + \frac{1}{n})x$, $x \ge 0$. Then, by [9, p. 255, Theorem 2.1] we have

$$|(S_n^\alpha f)(x) - f(x)|\ \le\ \left(1 + \frac{1}{2}\left(\alpha + \frac{1}{n}\right)\cdot\frac{x}{h^2}\right)\ \omega^2\left(f,h\right).$$

Putting here $h = \sqrt{(\alpha + 1/n)\ x/2}$ we obtain (4). Specifically we have

$$|(L_n f)(x) - f(x)|\ \le\ 2\ \omega^2(f, \sqrt{x/n}).$$

Thus the theorem is proved.

Let $f \in C_B[0,\infty)$ and $\beta \in (0,1]$. Then the Lipschitz - type maximal function of order β of f is defined as

$$f_\beta^\sim(x)\ =\ \sup_{\substack{t\ne x\\ t\in[0,\infty)}} \frac{|f(x) - f(t)|}{|x-t|^\beta},\qquad x \in [0,\infty).$$

Moreover, we define for $f \in C_B[0,\infty)$, $\beta \in (0,1]$ and $h > 0$ the following kind of generalized Lipschitz - type maximal function of order β and step - size h,

$$f_{\beta,h}^\sim(x)\ =\ \sup_{\substack{t\ne x\\ t\in[0,\infty)}} \frac{|\Delta_h^1(f,x) - \Delta_h^1(f,t)|}{|x-t|^\beta},\qquad x \in [0,\infty),$$

where $\Delta_h^1(f,x) = f(x+h) - f(x)$, $x \in [0,\infty)$, $h > 0$. Then, by standard method [5] we obtain the following result:

Theorem 2. *Let $f \in C_B[0,\infty)$ and $\beta \in (0,1]$. Then for all $x \in [0,\infty)$ and all $h > 0$ we have the inequalities*

a) $|(S_n^\alpha f)(x) - f(x)| \;\le\; f_\beta^\sim(x) \cdot (S_n^\alpha(\cdot - x)^\beta)(x);$

b) $|(S_n^\alpha f)(x) - f(x)| \;\le\; f_\beta^\sim(x) \cdot (2x/n)^{\beta/2};$

c) $|(S_n^\alpha f)(x) - f(x)| \;\le\; \left\{\frac{1}{h}\int_0^h f_{\beta,s}^\sim(x)\,ds\right\}(S_n^\alpha(\cdot - x)^\beta)(x) \;+\; \left\{\frac{1}{h} f_{\beta,h}^\sim(x)\right\} \cdot \frac{1}{1+\beta} \cdot$
$\cdot\, (S_n^\alpha(\cdot - x)^{1+\beta}, x);$

d) $|(S_n^\alpha f)(x) - f(x)| \;\le\; \left\{\frac{1}{h}\int_0^h f_{\beta,s}^\sim(x)\,ds\right\}(2x/n)^{\beta/2} \;+\; \left\{\frac{1}{h} f_{\beta,h}^\sim(x)\right\} \cdot \frac{1}{1+\beta} \cdot$
$\cdot\, (2x/n)^{(1+\beta)/2}.$

To establish the saturation result for S_n^α we use the following Voronovskaja - type formula:

Theorem 3. *Let $f \in C[0,\infty)$ be twice differentiable at some point $x > 0$ and let us assume that $f(t) = O(t^2)$. If $\alpha = \alpha(n)$ then*

$$\lim_{n\to\infty} n((S_n^\alpha f)(x) - f(x)) \;=\; \begin{cases} \frac{x}{2}\, f''(x), & \textit{for} \quad \alpha = o(n^{-1}) \\ x\, f''(x), & \textit{for} \quad \alpha = n^{-1}. \end{cases} \tag{5}$$

Proof. We obtain formula (5) following the proof of [1, Theorem 4]. Indeed, by Taylor's expansion

$$f\left(\frac{k}{n}\right) - f(x) \;=\; \left(\frac{k}{n} - x\right) f'(x) + \left(\frac{k}{n} - x\right)^2 \left(\frac{1}{2} f''(x) + \varepsilon\left(\frac{k}{n} - x\right)\right)$$

we obtain

$$\begin{aligned}(S_n^\alpha f)(x) - f(x) \;&=\; f'(x)(S_n^\alpha(e_1 - xe_0))(x) + \frac{1}{2} f''(x)(S_n^\alpha(e_1 - xe_0)^2)(x) + \\ &+\; (S_n^\alpha((e_1 - xe_0)^2 \varepsilon))(x),\end{aligned}$$

where ε is bounded and $\lim_{t\to 0} \varepsilon(t) = 0$. Because S_n^α leaves linear functions invariant we have

$$(S_n^\alpha f)(x) - f(x) \;=\; \frac{1}{2} f''(x)(S_n^\alpha(e_1 - xe_0)^2)(x) + (S_n^\alpha((e_1 - xe_0)^2 \varepsilon))(x). \tag{6}$$

Recalling the Cauchy - Schwarz inequality we obtain

$$\begin{aligned}(S_n^\alpha((e_1-xe_0)^2\varepsilon))(x) &\le (S_n^\alpha(e_1-xe_0)^2)(x)\ (S_n^\alpha((e_1-xe_0)^2\varepsilon))(x)\\ &\le \|\varepsilon^2\| \left(\alpha+\frac{1}{n}\right)^2 x^2.\end{aligned}$$

But $\alpha = o(n^{-1})$ or $\alpha = n^{-1}$ therefore

$$\lim_{n\to\infty} n(S_n^\alpha((e_1-xe_0)^2\varepsilon))(x) = 0.$$

Hence we conclude that (6), $\alpha = o(n^{-1})$ or $\alpha = n^{-1}$ and $(S_n^\alpha(e_1-xe_0)^2)(x) = \left(\alpha+\frac{1}{n}\right)x$ lead us to the asymptotic formula (5).

The saturation result is as follows:

Theorem 4. *Let $f \in C[0,\infty)$, $f(x) = O(x^2)$ and $\alpha = \alpha(n)$ such that $\alpha = o(n^{-1})$ or $\alpha = n^{-1}$. If $(S_n^\alpha f)(x) - f(x) = o_x(x/n)$ $(x \ge 0,\ n\to\infty)$ then f is a linear function; furthermore*

$$|(S_n^\alpha f)(x) - f(x)| \le M\cdot\frac{x}{n} \qquad (x\ge 0,\ n = 1,2,\ldots)$$

holds if and only if f has a derivative belonging to Lip 1, where

$$Lip\ 1 = \{\ f\ :\ |f(x+h)-f(x)| \le K_f h,\ x\ge 0,\ h>0\ \}.$$

Proof. By [4, Theorem 5.1] we have that f is locally and hence globally linear. Furthermore, we have $(S_n^\alpha(e_1-xe_0)^2)(x) = (\alpha+\frac{1}{n})x$ and the proofs of [4, Theorem 5.1] and [4, Theorem 5.4] hold for S_n^α on every finite interval $[a,b] \subseteq [0,\infty)$ and $(S_n^\alpha f)(x) - f(x) = O(x/n)$ implies that f has a derivative which is absolutely continuous on every interval $(a,b) \subseteq [0,\infty)$. But, in view of Theorem 3 we have $\lim_{n\to\infty} (n/x)\,((S_n^\alpha f)(x) - f(x)) = f''(x)/2$ or $\lim_{n\to\infty} (n/x)\,((S_n^\alpha f)(x) - f(x)) = f''(x)$ at every point x, where $f''(x)$ exists. So $(S_n^\alpha f)(x) - f(x) = O(x/n)$ implies $f''(x) = O(1)$ and this is the same as $f' \in Lip\ 1$.

The reverse statement follows from Theorem 1 since $f' \in Lip\ 1$ implies $\omega^2(f,\delta) = O(\delta^2)$. Thus the theorem is proved.

In [7, p. 244, Theorem 4.2] is established the inequality $f(x) \le (S_n^\alpha f)(x)$, $x \ge 0$ for a convex function $f \in C_B[0,\infty)$. The next theorem gives a similar result without use the evaluation of the remainder term.

Theorem 5. *Let $f \in C_B[0, \infty)$ be a convex function.*

Then $f(x) \leq (S_n f)(x) \leq (S_n^\alpha f)(x)$ *for all* $x \geq 0$.

Proof. The first inequality is known [2]. For the second inequality we consider the following Taylor's expansion:

$$(S_n f)(t) = (S_n f)(x) + (t-x)(S_n f)'(x) + \int_x^t (t-u)(S_n f)''(u)\, du.$$

Hence, by [7, p. 240, Theorem 2.8] we obtain

$$\begin{aligned}
&(S_n^\alpha f)(x) - (S_n f)(x) = \\
&= \frac{1}{\Gamma\left(\frac{x}{\alpha}\right)} \cdot \int_0^\infty e^{-t}\, t^{\frac{x}{\alpha}-1}\, (S_n f)(\alpha t)\, dt - (S_n f)(x) \\
&= \frac{1}{\alpha\, \Gamma\left(\frac{x}{\alpha}\right)} \cdot \int_0^\infty e^{-\frac{t}{\alpha}} \left(\frac{t}{\alpha}\right)^{\frac{x}{\alpha}-1} [\, (S_n f)(t) - (S_n f)(x)\,]\, dt \\
&= \frac{1}{\alpha\, \Gamma\left(\frac{x}{\alpha}\right)} \cdot \int_0^\infty e^{-\frac{t}{\alpha}} \left(\frac{t}{\alpha}\right)^{\frac{x}{\alpha}-1} \Big[(t-x)(S_n f)'(x) + \\
&\qquad + \int_x^t (t-u)(S_n f)''(u)\, du \Big]\, dt.
\end{aligned}$$

But $(S_n^\alpha e_1)(x) = e_1(x)$, therefore

$$\begin{aligned}
&(S_n^\alpha f)(x) - (S_n f)(x) = \\
&= \frac{1}{\alpha\, \Gamma\left(\frac{x}{\alpha}\right)} \cdot \left\{ \int_0^x e^{-\frac{t}{\alpha}} \left(\frac{t}{\alpha}\right)^{\frac{x}{\alpha}-1} \cdot \left[\int_t^x (u-t)(S_n f)''(u)\, du \right] dt + \right. \\
&\qquad \left. + \int_x^\infty e^{-\frac{t}{\alpha}} \left(\frac{t}{\alpha}\right)^{\frac{x}{\alpha}-1} \cdot \left[\int_x^t (t-u)(S_n f)''(u)\, du \right] dt \right\}.
\end{aligned}$$

On the other hand

$$(S_n f)''(x) = e^{-nx} \cdot n^2 \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \cdot \Delta^2_{1/n}(f, \frac{k}{n}) \geq 0$$

for the convex function f [2, *p.* 136]. So $(S_n^\alpha f)(x) \geq (S_n f)(x)$.

Remark. The inequality $f(x) \leq (S_n^\alpha f)(x)$ can be proved by Jensen's inequality as well.

Indeed, let

$$s_{n,k}(x, \alpha) = (1 + n\alpha)^{-x/\alpha} \cdot \left(\alpha + \frac{1}{n}\right)^{-k} \cdot \frac{x(x+\alpha)\ldots(x+(k-1)\alpha)}{k!}, \quad k \geq 0.$$

Then, by [7, p. 239, Theorem 2.3] we have $\sum_{k=0}^{\infty} s_{n,k}(x,\alpha) = 1$ and

$$\sum_{k=0}^{\infty} \frac{k}{n} \; s_{n,k}(x,\alpha) \;=\; x, \qquad x \geq 0. \tag{7}$$

Using Jensen's inequality we obtain

$$\sum_{k=0}^{m} s_{n,k}(x,\alpha)\; f\left(\frac{k}{n}\right) + \left[\sum_{k\geq m+1} s_{n,k}(x,\alpha)\right] f(0) \;\geq\; f\left(\sum_{k=0}^{m} s_{n,k}(x,\alpha)\cdot\frac{k}{n}\right).$$

Hence, by continuity of f and (7) we obtain

$$(S_n^{\alpha} f)(x) \;=\; \sum_{k=0}^{\infty} s_{n,k}(x,\alpha)\; f\left(\frac{k}{n}\right) \;\geq\; f\left(\sum_{k=0}^{\infty} s_{n,k}(x,\alpha)\cdot\frac{k}{n}\right) \;=\; f(x).$$

This completes the proof.

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Babeş - Bolyai University, Department of Mathematics and Computer Science, str. M. Kogălniceanu 1, 3400 Cluj, Romania
E-mail address: `fzoltanmath.ubbcluj.ro`