STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 4, December 2001

LACUNARY STRONG A-CONVERGENCE WITH RESPECT TO A MODULUS

TUNAY BILGIN

Abstract. The definition of lacunary strong convergence with respect to a modulus is extended to a definition of lacunary strong A-convergence with respect to a modulus when $A = (a_{ik})$ is an infinite matrix of complex numbers. We study some connections between lacunary strong A-convergence with respect to a modulus and lacunary A-statistical convergence.

1. **Introduction**

The notion of modulus function was introduced by Nakano [11]. We recall that a modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x + y) \leq f(x) + f(y)$ for $x, y \geq 0$,

(iii) f is increasing and

(iv) f is continuous from the right at 0. It follows that f must be continuous on $[0, \infty)$.

Connor [2], Esi [3], Kolk [8], Maddox [9], [10], Öztürk and Bilgin [12], Pehlivan and Fisher [13], Ruckle [14] and others used a modulus function to construct sequence spaces.

Following Freedman et al. [4], we call the sequence $\theta = (k_r)$ lacunary if it is an increasing sequence of integers such that $k_0 = 0$, $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $q_r = k_r / k_{r-1}$. These notations will be used throughout the paper. The sequence space of lacunary

1991 *Mathematics Subject Classification.* 40A05, 40F05.

Key words and phrases. lacunary sequence, modulus function, statistical convergence.

strongly convergent sequences N_θ was defined by Freedman et al. [4], as follows:

$$N_\theta = \left\{ x = (x_i) : \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} |x_i - s| = 0 \text{ for some } s \right\}.$$

Recently, the concept of lacunary strongly convergence was generalized by Pehlivan and Fisher [13] as below:

$$N_\theta(f) = \left\{ x = (x_i) : \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} f(|x_i - s|) = 0 \text{ for some } s \right\}.$$

Let $A = (a_{ik})$ be an infinite matrix of complex numbers. We write $Ax = (A_i(x))$ if $A_i(x) = \sum_{k=1}^{\infty} a_{ik}x_k$ converges for each i.

The purpose of this paper is to introduce and study a concept of lacunary strong A-convergence with respect to a modulus.

2. $N_\theta(A, f)$ Convergence

Definition. Let $A = (a_{ik})$ be an infinite matrix of complex numbers and f be a modulus. We define

$$N_\theta(A, f) = \left\{ x = (x_i) : \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} f(|A_i(x) - s|) = 0 \text{ for some } s \right\},$$

$$N_\theta^0(A, f) = \left\{ x = (x_i) : \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} f(|A_i(x)|) = 0 \right\}.$$

A sequence $x = (x_k)$ is said to be lacunary strong A-convergent to a number s with respect to a modulus if there is a complex number s such that $x \in N_\theta(A, f)$. Note that, if we put $f(x) = x$, then $N_\theta(A, f) = N_\theta(A)$ and $N_\theta^0(A, f) = N_\theta^0(A)$. If $x \in N_\theta(A)$, we say that x is lacunary strong A-convergent to s. If x is lacunary strong A-convergent to the value s with respect to a modulus f, then we write $x_i \to s(N_\theta(A, f))$. If $A = I$ unit matrix, we write $N_\theta(f)$ and $N_\theta^0(f)$ for $N_\theta(A, f)$ and $N_\theta^0(A, f)$, respectively. Hence $N_\theta(f)$ is the same as the space $N_\theta(f)$ of Pehlivan and Fisher [13].

$N_\theta(A, f)$ and $N_\theta^0(A, f)$ are linear spaces. We consider only $N_\theta^0(A, f)$. Suppose that $x, y \in N_\theta^0(A, f)$ and a, b are in C, the complex numbers. Then there exist integers

T_a and T_b such that $|a| \leq T_a$ and $|b| \leq T_b$. We therefore have

$$h_r^{-1} \sum_{i \in I_r} f(|aA_i(x) + bA_i(y)|) \leq T_a h_r^{-1} \sum_{i \in I_r} f(|A_i(x)|) + T_b h_r^{-1} \sum_{i \in I_r} f(|A_i(y)|).$$

This implies that $ax + by \in N_\theta^0(A, f)$.

Now we give relation between lacunary strong A-convergence and lacunary strong A-convergence with respect to a modulus.

Theorem 1. *Let f be any modulus. Then $N_\theta(A) \subseteq N_\theta(A, f)$ and $N_\theta^0(A) \subseteq N_\theta^0(A, f)$.*

Proof. We consider $N_\theta(A) \subseteq N_\theta(A, f)$ only. Let $x \in N_\theta(A)$ and $\varepsilon \succ 0$. We choose $0 < \delta < 1$ such that $f(u) < \varepsilon$ for every u with $0 \leq u \leq \delta$. We can write

$$h_r^{-1} \sum_{i \in I_r} f(|A_i(x) - s|) = h_r^{-1} \sum_{1} f(|A_i(x) - s|) + h_r^{-1} \sum_{2} f(|A_i(x) - s|)$$

where the first summation is over $|A_i(x) - s| \leq \delta$ and the second over $|A_i(x) - s| \succ \delta$. By definition of f, we have

$$h_r^{-1} \sum_{i \in I_r} f(|A_i(x) - s|) \leq \varepsilon + 2f(1)\delta^{-1} h_r^{-1} \sum_{i \in I_r} |A_i(x) - s|.$$

Therefore $x \in N_\theta(A, f)$.

Theorem 2. *Let f be any modulus. If $\lim_{t \to \infty} \dfrac{f(t)}{t} = \beta \succ 0$, then $N_\theta(A) = N_\theta(A, f)$.*

Proof. If $\lim_{t \to \infty} \dfrac{f(t)}{t} = \beta \succ 0$, then $f(t) \geq \beta t$ for all $t \succ 0$. Let $x \in N_\theta(A, f)$. Clearly,

$$h_r^{-1} \sum_{i \in I_r} f(|A_i(x) - s|) \geq h_r^{-1} \sum_{i \in I_r} \beta |A_i(x) - s| = \beta h_r^{-1} \sum_{i \in I} |A_i(x) - s|,$$

therefore $x \in N_\theta(A)$. By using Theorem 1 the proof is complete.

We now give an example to show that $N_\theta(A) \neq N_\theta(A, f)$ in the case when $\beta = 0$. Consider $A = I$ and the modulus $f(x) = \sqrt{x}$. In the case $\beta = 0$, define x_i to be h_r at the first term in I_r for every r and $x_i = 0$ otherwise. Then we have

$$h_r^{-1} \sum_{i \in I_r} f(|A_i(x)|) = h_r^{-1} \sum_{i \in I_r} \sqrt{|x_i|} = h_r^{-1} \sqrt{|h_r|} \to 0 \text{ as } r \to \infty$$

and so $x \in N_\theta(A, f)$. But $h_r^{-1} \sum_{i \in I_r} |A_i(x)| = h_r^{-1} \sum_{i \in I_r} |x_i| = h_r^{-1} h_r \to 1$ as $r \to \infty$ and so $x \notin N_\theta(A)$.

Theorem 3. *Let f be any modulus. Then* *(i) For* $\liminf q_r \succ 1$ *we have* $w(A,f) \subseteq N_\theta(A,f)$.

(ii) For $\limsup q_r \prec \infty$ *we have* $N_\theta(A,f) \subseteq w(A,f)$.

(iii) $w(A,f) = N_\theta(A,f)$ *is* $1 \succ \liminf_r q_r \leq \limsup_r q_r \prec \infty$,

where $w(A,f) = \left\{ x = (x_i) : \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} f(|A_i(x) - s|) = 0 \text{ for some } s \right\}$ *(see, Esi [3]).*

Proof. (i) Let $x \in w(A,f)$ and $\liminf q_r \succ 1$. There exist $\delta \succ 0$ such that $q_r = (k_r/k_{r-1}) \geq 1+\delta$ for sufficiently large r. We have, for sufficiently large r, that $(h_r/k_r) \geq \delta/(1+\delta)$ and $(k_r/h_r) \leq (1+\delta)/\delta$. Then

$$\begin{aligned} k_r^{-1} \sum_{i-1}^{k_r} f(|A_i(x) - s|) &\geq k_r^{-1} \sum_{i\in I_r} f(|A_i(x) - s|) \\ &= (h_r/k_r) h_r^{-1} \sum_{i\in I_r} f(|A_i(x) - s|) \\ &\geq \delta/(1+\delta) h_r^{-1} \sum_{i\in I_r} f(|A_i(x) - s|) \end{aligned}$$

which yields that $x \in N_\theta(A,f)$.

(ii) If $\limsup q_r \prec \infty$ then there exists $K \succ 0$ such that $q_r \prec K$ for every r. Now suppose that $x \in N_\theta(A,f)$ and $\varepsilon \succ 0$. There exists m_0 such that for every $m \geq m_0$,

$$H_m = h_m^{-1} \sum_{i\in I_m} f(|A_i(x) - x|) \prec \varepsilon.$$

We can also find $T \succ 0$ such that $H_m \leq T$ for all m. Let n be any integer with $k_r \geq n \succ k_{r-1}$. Now write

$$n^{-1} \sum_{i=1}^{n} f(|A_i(x) - s|) \leq k_r^{-1} \sum_{i=1}^{k_r} f(|A_i(x) - s|)$$

$$= k_{r-1}^{-1} \left(\sum_{m=1}^{m_0} + \sum_{m=m_0+1}^{k_r} \right) \sum_{i\in I_m} f(|A_i(x) - s|)$$

$$= k_{r-1}^{-1} \sum_{m=1}^{m_0} \sum_{i\in I_m} f(|A_i(x) - s|) + k_{r-1}^{-1} \sum_{m=m_0+1}^{k_r} \sum_{i\in I_m} f(|A_i(x) - s|)$$

$$\leq k_{r-1}^{-1} \sum_{m=1}^{m_0} \sum_{i\in I_m} f(|A_i(x) - s|) + \varepsilon(k_r - k_{m_0}) k_{r-1}^{-1}$$

$$= k_{r-1}^{-1} (h_1 H_1 + h_2 H_2 + \cdots + h_{m_0} H_{m_0}) + \varepsilon(k_r - k_{m_0}) k_{r-1}^{-1}$$

$$\leq k_{r-1}^{-1} \left(\sup_{1\leq i\leq m_0} H_i k_{m_0} \right) + \varepsilon K \prec k_{r-1}^{-1} k_{m_0} T + \varepsilon K$$

from which we deduce that $x \in w(A, f)$. (iii) follows from (i) and (ii).

The next result follows from Theorem 2 and 3.

Theorem 4. *Let f be any modulus. If* $\lim_{t\to\infty} \dfrac{f(t)}{t} = \beta \succ 0$ *and* $l \prec \liminf_r q_r \leq \limsup_r q_r \prec \infty$, *then* $N_\theta(A) = w(A, f)$.

3. **Lacunary A-statistical convergence**

The notation of statistical convergence was given in earlier works [1], [4], [6], [15] and [16]. Recently, Fridy and Orhan [7] introduced the concept of lacunary statistical convergence:

Let θ be a lacunary sequence. Then a sequence $x = (x_k)$ is said to be lacunary statistically convergent to a number s if for every $\varepsilon \succ 0$, $\lim_{r\to\infty} h_r^{-1}|K_\theta(\varepsilon)| = 0$, where $|K_\theta(\varepsilon)|$ denotes the number of elements in $K_\theta(\varepsilon) = \{i \in I_r : \ |x_i - s| \geq \varepsilon\}$. The set of all lacunary statistical convergent sequences is denoted by S_θ.

Let $A = (a_{ik})$ be an infinire matrix of complex numbers. Then a sequence $x = (x_k)$ is said to be lacunary A-statistically convergent to a number s if for every $\varepsilon \succ 0$, $\lim_{r\to\infty} h_r^{-1}|KA_\theta(\varepsilon)| = 0$, where $|KA_\theta(\varepsilon)|$ denotes the number of element in $KA_\theta(\varepsilon) = \{i \in I : \ |A_i(x) - s| \geq \varepsilon\}$. The set of all lacunary A-statistical convergent sequences is denoted by $S_\theta(A)$.

The following Theorem gives the relation between of the lacunary A-statistical convergence and lacunary strongly A-convergence.

Let $I_r^1 = \{i \in I_r : \ |A_i(x) - s| \geq \varepsilon\} = KA_\theta(\varepsilon)$ and $I_r^2 = \{i \in I_r : \ |A_i(x) - s| \prec \varepsilon\}$.

Theorem 5. *Let A be a limitation method, then*

(i) $x_i \to s(N_\theta(A))$ implies $x_i \to s(S_\theta(A))$.

(ii) x is bounded and $x_i \to s(S_\theta(A))$ implys $x_i \to s(N_\theta(A))$.

(iii) $S_\theta(A) = N_\theta(A)$ is x is bounded.

Proof. (i) If $\varepsilon \succ 0$ and $x_i \to s(N_\theta(A))$ we can write

$$h_r^{-1} \sum_{i\in I_r} |A_i(x) - s| \geq h_r^{-1}|KA_\theta(\varepsilon)|\varepsilon.$$

It follows that $x_i \to s(S_\theta(A))$. Note that in this part of the proof we do not need the limitation method of A.

(ii) Suppose that x is lacunary A-statistical convergent to s. Since x is bounded and A is limitation method, there is a constant $T > 0$ such that $|A_i(x) - s| \leq T$ for all i. Therefore we have, for every $\varepsilon \succ 0$, that

$$h_r^{-1} \sum_{i \in I_r} |A_i(x) - s| \leq h_r^{-1} \sum_{i \in I_r^1} |A_i(x) - s| + h_r^{-1} \sum_{i \in I_r^2} |A_i(x) - s| \leq T h_r^{-1} |KA_\theta(\varepsilon)| + \varepsilon.$$

Taking the limit as $\varepsilon \to 0$, the result follows. (iii) follows from (i) and (ii).

Now we give the relation between of the lacunary A-statistical convergence and lacunary strongly A-convergence with respect to modulus.

Theorem 6. *(i) For any modulus f, $x_i \to s(N_\theta(A, f))$ implies $x_i \to s(S_\theta(A))$.*

(ii) f is bounded and $x_i \to s(S_\theta(A))$ imply $x_i \to s(N_\theta(A, f))$.

(iii) $S_\theta(A) = N_\theta(A, f)$ if f is bounded.

Proof. (i) Let f be any modulus. If $\varepsilon \succ 0$ and $x_i \to s(N_\theta(A, f))$ we can write

$$h_r^{-1} \sum_{i \in I_r} f(|A_i(x) - s|) \geq h_r^{-1} \sum_{i \in I_r^1} f(|A_i(x) - s|) \succ h_r^{-1} |KA_\theta(\varepsilon)| f(\varepsilon).$$

It follows that $x_i \to s(S_\theta(A))$.

(ii) Suppose that f is bounded. Since f is bounded, there exists an integer T such that $f(x) \leq T$ for all $x \geq 0$. We see that

$$\begin{aligned} h_r^{-1} \sum_{i \in I_r} f(|A_i(x) - s|) &\leq h_r^{-1} \sum_{i \in I_r^1} f(|A_i(x) - s|) + h_r^{-1} \sum_{i \in I_r^2} f(|A_i(x) - s|) \\ &\leq T h_r^{-1} |KA_\theta(\varepsilon)| + f(\varepsilon). \end{aligned}$$

Since f is continuous and $x_i \to s(S_\theta(A))$, it follows from $\varepsilon \to 0$ that $x_i \to s(N_\theta(A, f))$. (ii) follows from (i) and (ii).

As an example to show that $S_\theta(A) \neq N_\theta(A, f)$ when f is unbounded, consider $A = I$. Since f is unbounded, there exists a positive sequence $0 \prec y_1 \prec y_2 \prec \ldots$ such that $f(y_i) \geq h_i$. Define the sequence $x = (x_i)$ by putting $x_{k_i} = y_i$ for $i = 1, 2, \ldots$ and $x_i = 0$ otherwise. We have $x \in S_\theta(A)$, but $x \notin N_\theta(A, f)$.

Finally we consider the case when $x_k \to s$ implies $x_k \to s(N_\theta(A, f))$.

Lemma 7. ([6]) *If $\liminf q_r \succ 1$ then $x_i \to s(S)$ implies $x_i \to s(S_\theta)$.*

Theorem 8. *Let* $\liminf q_r \succ 1$*, A is regular and f is bounded. Then* $x_i \to s$ *implies* $x_i \to s(N_\theta(A, f))$.

Proof. Let $x_i \to s$. By regularity of A and definition of statistical convergence we have $A_i(x) \to s(S)$. Since $\liminf q_r \succ 1$ it follows lemma 7 that $A_i(x) \to s(S_\theta)$ i.e. $x_i \to s(S_\theta(A))$. Thus, using Theorem 6, we have $x_i \to s(N_\theta(A, f))$.

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Department of Mathematics, University of 100.Yil, Van Turkey