STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 4, December 2001

INTERPOLATION RESULTS FOR SOME CLASSES OF ABSOLUTELY SUMMING OPERATORS

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Abstract. K. Miyazaki, [9], has introduced the class of $(p, q; r)$-absolutely summing operators, which generalize the class of (p, q)-absolutely summing operators, introduced by Mitiagin and Pełczynski in 1966.

We establish an interpolation result for $(p, q; r)$ −absolutely summing operators and also for some other operator classes which generalize Miyazaki"s classes.

1. **Introduction**

The interpolation properties of the $p-$summable and the (p, q)-absolutely summing operators are well known. Miyazaki has extended the result concerning the interpolation stability for $(p, q)-$absolutely summing operators to the more general ideal of $(p, q; r)$-absolutely summing operators, which he introduced [9]. In this paper we will look at his result, because it relies on the presumption that the ideal of $(p, q; r)$-absolutely summing operators is normed, which in general does not happen, this ideal being only quasi-normed. N. Tita [11], [12] has introduced and studied ideals of operators which are (Φ, Ψ)-absolutely summing, where Φ and Ψ are symmetric norming functions, and which are more general than the (p, q)-absolutely summing operators and the largest part of the ideals studied by Miyazaki. Due to the nonlinearity of the symmetric norming functions, nothing could be ascertained regarding the interpolation properties of these ideals of operators. For this reason we ask the question of existence of ideals of operators more general than those of Miyazaki, and which still satisfy the stability result proved by him. In order to answer to the above question we construct a class of absolutely summing opetators, which is based on the Lorentz-Zygmund spaces of sequences.

Key words and phrases. p-absolutely summing operators, p-q absolutely summing operators, symmetric norming function, Lorentz-Zygmund sequence ideals.

The present paper is a revised and extended version of [1]. This revision became necessary as we had not, at the time of writing [1], been aware of Myazaki"s work, and we realized that the class we had introduced was not satisfactorily motivated nor exhaustively treated.

2. **Preliminaries**

We first introduce some notation and recall a few known results. Throughout the paper $\mathbb{N}$ denotes the set of all positive integers, while E, F are Banach spaces over Γ, where Γ is the real or the complex field. By $\mathcal{F}(E)$ we denote a finite set of vectors $x_1, ..., x_n$ in E. We denote

$$L(E,F) := \{T : E \to F : \ T \text{ is linear and bounded}\},$$

and we let E^* be the dual space, $E^* = L(E, \Gamma)$. By U_E we denote the unit ball $\{x \in E : \|x\| \leq 1\}$. For $a \in E^*$ and $x \in E$, let $\langle x, a\rangle := a(x)$. We denote by l_∞ the set of all scalar sequences, $\{x_n\}_n$, with the property $\|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n| < \infty$, and by c_0 the set of all scalar sequences, $\{x_n\}_n$, with the property $\lim_{n\to\infty} |x_n| = 0$. For $0 < p < \infty$, we let l_p denote the set of all scalar sequences $\{x_n\}_n$ such that $\|x\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} < \infty$.

The operator classes, which are the subject of this article, are closely related to some vector-valued sequence spaces. For this reason we shall recall here a few definitions and results about these spaces.

Definition 1. ([5]) Let $1 \leq p \leq \infty$. The vector sequence $\{x_n\}_n$ in E is strongly $p-$summable if the corresponding scalar sequence $\{\|x_n\|\}_n$ is in l_p. We denote by $l_p^{strong}(E)$ the set of all such sequences in E.

It is clearly a vector space under pointwise operations, and a natural norm is given by $\|\{x_n\}\|_p^{strong} := \left(\sum_{n=1}^{\infty} \|x_n\|^p\right)^{\frac{1}{p}}$, respectively $\|\{x_n\}\|_\infty^{strong} := \sup_n \|x_n\|$.

Definition 2. ([5]) Let $1 \leq p \leq \infty$. The vector sequence $\{x_n\}_n$ in E is weakly $p-$summable if the scalar sequences $\{|\langle x^*, x_n\rangle|\}_n$ are in l_p for every $x^* \in E^*$. We denote by $l_p^{weak}(E)$ the set of all such sequences in E.

It is clearly a vector space under pointwise operations, and a norm is given by $\|\{x_n\}\|_p^{weak} := \sup_{x^* \in U_{E^*}} \left(\sum_{n=1}^{\infty} |\langle x^*, x_n \rangle|^p \right)^{\frac{1}{p}}$, respectively $\|\{x_n\}\|_\infty^{weak} := \sup_{x^* \in U_{E^*}} \sup_n |\langle x^*, x_n \rangle| = \sup_n \|x_n\| = \|\{x_n\}\|_\infty^{strong}$.

Definition 3.([13]) For $x = \{x_n\}_n \in l_\infty$, let

$$s_n(x) := \inf \{\sigma \geq 0 : card\{i : |x_i| \geq \sigma\} < n\}.$$

Proposition 4. ([13]) The numbers $s_n(x)$ have the following properties:

(1) . $\|x\|_\infty = s_1(x) \geq s_2(x) \geq ... \geq 0$, for all $x = \{x_n\}_n \in l_\infty$;

(2) . $s_{n+m-1}(x+y) \leq s_n(x) + s_m(y)$, for all $x = \{x_i\}_i \in l_\infty, y = \{y_i\}_i \in l_\infty$, and $n, m \in \{1, 2, ...\}$, where $x + y = \{x_i + y_i\}_i$;

(3) . $s_{n+m-1}(x \cdot y) \leq s_n(x) \cdot s_m(y)$, for all $x = \{x_i\}_i \in l_\infty, y = \{y_i\}_i \in l_\infty$, and $n, m \in \{1, 2, ...\}$, where $x \cdot y = \{x_i \cdot y_i\}_i$;

(4) . If $x = \{x_m\}_m \in l_\infty$ and $card\{m : x_m \neq 0\} < n$ then $s_n(x) = 0$.

If the sequence $x = \{x_n\}_n \in l_\infty$ is ordered such that $|x_n| \geq |x_{n+1}|$, for any natural n, then $s_n(x) = |x_n|$, [13].

Definition 5. (Lorentz sequence spaces) ([9]) Let $1 \leq p \leq \infty$, $1 \leq q < \infty$, or $1 \leq p \leq \infty$, $q = \infty$. The vector sequence $\{x_n\}_n$ in E is strongly (p, q) −summable if $\sum_{n=1}^{\infty} \left[i^{\frac{1}{p} - \frac{1}{q}} \cdot s_n(\|x\|) \right]^q$ is finite, respectively $\sup_n i^{\frac{1}{p}} \cdot s_n(\|x\|)$ is finite, where

$$s_n(\|x\|) := s_n\left(\{\|x_i\|_E\}_i\right).$$

The space of all such sequences in E will be called the Lorentz sequence space and will be denoted by $l_{p,q}^{strong}(E)$. In particular, if $E = \Gamma$, then $l_{p,q}^{strong}(\Gamma)$ is denoted $l_{p,q}$.

It is clear that $l_{p,q}^{strong}(E)$ is a vector space under pointwise operations, and a natural quasi-norm is given by

$$\|\{x_n\}\|_{p,q}^{strong} := \left(\sum_{n=1}^{\infty} \left[i^{\frac{1}{p} - \frac{1}{q}} \cdot s_n(\|x\|) \right]^q \right)^{\frac{1}{q}},$$

respectively

$$\|\{x_n\}\|_{p,\infty}^{strong} := \sup_n i^{\frac{1}{p}} \cdot s_n(\|x\|).$$

It is important for our future considerations to recall the **lexicografic order** of the Lorentz spaces.

Proposition 6. ([7], [9]) (1) Let $1 \leq p < \infty$, $1 \leq q < q_1 \leq \infty$. Then $l_{p,q}^{strong}(E) \subset l_{p,q_1}^{strong}(E)$ and for every $\{x_i\}_i \in l_{p,q}^{strong}(E)$,

$$\|\{x_n\}\|_{p,q_1}^{strong} \leq c(p,q,q_1) \cdot \|\{x_n\}\|_{p,q}^{strong}.$$

(2) Let $1 \leq p < p_1 \leq \infty$, $1 \leq q, q_1 \leq \infty$. Then $l_{p,q}^{strong}(E) \subset l_{p_1,q_1}^{strong}(E)$ and, for every $\{x_i\}_i \in l_{p,q}^{strong}(E)$,

$$\|\{x_n\}\|_{p_1,q_1}^{strong} \leq \overline{c}(p,p_1,q,q_1) \cdot \|\{x_n\}\|_{p,q}^{strong}.$$

We now recall, from [6], some basic facts about the classical real interpolation method, called the K-method. An interpolation method is a method of constructing interpolation spaces from a given couple of spaces. For the reader interested in finding an introduction to interpolation theory we recommend, for example, [2], [6], [15].

We consider couples (A_0, A_1) of topological vector spaces A_0, A_1, which are both continuously embedded in a topological vector space $\mathcal{A}$. We denote this by $A_i \hookrightarrow \mathcal{A}$, $i = 1, 2$ and we say that (A_0, A_1) is an interpolation couple.

If (A_0, A_1), (B_0, B_1) are two such couples with $A_0, A_1 \hookrightarrow \mathcal{A}$, $B_0, B_1 \hookrightarrow \mathcal{B}$ and if A and B are two other spaces with $A \hookrightarrow \mathcal{A}$ and $B \hookrightarrow \mathcal{B}$ we say that A and B are interpolation spaces with respect to the couples (A_0, A_1) and (B_0, B_1) if the following interpolation property is fulfilled:

For every linear operator T such that $T : A_0 \to B_0$, $T : A_1 \to B_1$ it follows that $T : A \to B$.

Here we let the symbol $T : A \to B$ denote that the restriction to A of the linear operator T is continuous.

Let (A_0, A_1) be an interpolation couple of quasi-normed spaces. For every $a \in A_0 + A_1$ we define the functional

$$K(t, a, A_0, A_1) = K(t, a) = \inf_{a = a_0 + a_1} \left(\|a_0\|_{A_0} + t \cdot \|a_1\|_{A_1} \right),$$

where $a_i \in A_i, i = 0, 1$, and $0 < t < \infty$.

For $0 < \theta < 1$ and $0 < q \leq \infty$ the spaces

$$(A_0, A_1)_{\theta,q} := \left\{ a; a \in A_0 + A_1 : \left(\int_0^\infty \left[t^{-\theta} \cdot K(t, a) \right]^q \frac{dt}{t} \right) < \infty \right\},$$

if $q < \infty$, and

$$(A_0, A_1)_{\theta,\infty} := \left\{ a; a \in A_0 + A_1 : \sup_{t>0} \sup t^{-\theta} \cdot K(t, x) < \infty \right\}$$

with the quasi-norm

$$\|a\|_{(A_0,A_1)_{\theta,q}} := \left(\int_0^\infty \left[t^{-\theta} \cdot K(t, a) \right]^q \frac{dt}{t} \right)^{\frac{1}{q}},$$

respectively

$$\|a\|_{(A_0,A_1)_{\theta,\infty}} = \sup_{t>0} \sup t^{-\theta} \cdot K(t, x),$$

are interpolation spaces. We have the following fundamental interpolation theorem.

Theorem 7. ([6]) *If* (A_0, A_1), (B_0, B_1) *are two interpolation couples of quasi-normed spaces and if* T *is a linear operator such that* $T : A_0 \to B_0$, $T : A_1 \to B_1$ *are both bounded, having the quasi-norms bounded from above by* M_0 *and* M_1 *respectively, then* $T : (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q}$ *is also bounded, and its quasi-norm is bounded from above by* M *for which we have the so called convexity inequality* $M \leq M_0^{1-\theta} \cdot M_1^{\theta}$.

Theorem 8. ([13]) *Let* $1 \leq p_0 < p_1 < \infty$, $1 \leq q_0, q_1, q \leq \infty$, $0 < \theta < 1$. *If* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *then* $\left(l^{strong}_{p_0,q_0}(E), l^{strong}_{p_1,q_1}(E) \right)_{\theta,q} = l^{strong}_{p,q}(E)$. *Moreover, the quasi-norms on both sides are equivalent.*

We can now introduce some classes of absolutely summing operators.

Definition 9. ([5]) Let $1 \leq p < \infty$. An operator $T \in L(E, F)$ is called absolutely p**-**summing**,** we write $T \in \Pi_p(E, F)$, if there is a constant $c \geq 0$ such that

$$\left(\sum_{i=1}^n \|Tx_i\|^p \right)^{\frac{1}{p}} \leq c \cdot \sup_{a \in U_{E^*}} \left(\sum_{i=1}^n |\langle x_i, a \rangle|^p \right)^{\frac{1}{p}},$$

for every finite family of elements $x_1, ...x_n \in E$.

For $T \in \Pi_p(E, F)$ we define $\pi_p(T) := \inf c$, the infimum being taken over all constants $c \geq 0$ for which the above inequality holds.

Note that $\pi_p(\cdot)$ is a norm on the space of absolutely $p-$summing operators, [5], [10].

The most deep result concerning absolutely $p-$summing operators is given by the following statement called **the domination theorem.**

Theorem 10. ([5], [10]) *Let* $1 \leq p < \infty$, $T \in L(E, F)$ *and* K *be a weak*-compact norming subset of* U_{E^*}. *Then* $T \in \Pi_p(E, F)$ *if and only if there is a constant* c *and a* *regular probability measure* μ *on* K *such that*

$$\|Tx\| \leq c \cdot \left(\int_{U_{E^*}} \left(|\langle x, x^* \rangle| \right)^p d\mu \left(x^* \right) \right)^{\frac{1}{p}},$$

for every $x \in E,$ *and* $\pi_p(T) = \inf c.$

We conclude this section by recalling the definition of the $(p, q; r)$ −absolutely summing operators

Definition 11. ([9]) For $1 \leq p, q, r \leq \infty$ an operator $T \in L(E, F)$ is called $(p, q; r)$ −absolutely summing provided there exists a constant $c > 0$ such that

$$\|\{Tx_i\}\|_{p,q}^{strong} \leq c \cdot \|\{x_i\}\|_r^{weak}$$

for every $\{x_i\} \in \mathcal{F}(E)$. We denote by $\Pi_{p,q;r}(E, F)$ the space of such operators acting between E and F.

The smallest number c for which the above inequality holds is denoted by $\pi_{p,q;r}(T)$.

It is observed in [7] that $\pi_{p,q;r}(\cdot)$ is a quasi-norm on the space of $(p, q; r)$-absolutely summing operators.

3. Results

We are first concerned with the interpolation result for $(p, q; r)$ −absolutely summing operators established in [9], as we also indicated in the introduction.

Theorem 12. *Let* $1 \leq p_1 < p_2 < \infty$, $1 \leq q_1, q_2, q, r < \infty$ *and* $0 < \theta < 1$. *If* $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ *then* $\left(\Pi_{p_1,q_1;r}(E, F), \Pi_{p_2,q_2;r}(E, F) \right)_{\theta,q} \subset \Pi_{p,q;r}(E, F)$.

Proof. We shall use an idea owed to H. König, see Proposition 3 from [8], but first we must prove that $\left(\Pi_{p_1,q_1;r}(E, F), \Pi_{p_2,q_2;r}(E, F) \right)$ is an interpolation couple.

Let $T \in \Pi_{p_1,q_1;r}(E, F)$ and $\{x_i\}_i \in \mathcal{F}(E)$. It follows that there exists a constant $c > 0$ such that $\|\{Tx_i\}\|_{p_1,q_1}^{strong} \leq c \cdot \|\{x_i\}\|_r^{weak}$.

But we know that $\|\{Tx_n\}\|_{p_2,q_2}^{strong} \leq \overline{c} \cdot \|\{Tx_n\}\|_{p_1,q_1}^{strong}$. Thus we obtain $\|\{Tx_n\}\|_{p_2,q_2}^{strong} \leq \overline{c} \cdot \|\{Tx_n\}\|_{p_1,q_1}^{strong} \leq \widetilde{c} \cdot \|\{x_i\}\|_r^{weak}$. In conclusion $T \in \Pi_{p_2,q_2;r}(E, F)$ and $\Pi_{p_1,q_1;r}(E, F) \subset \Pi_{p_2,q_2;r}(E, F)$.

Let now $T \in \Pi_{p_2,q_2;r}(E, F)$ and take $\{x_i\}_{i=1}^n \in \mathcal{F}(E)$ with $\|\{x_i\}\|_r^{weak} = 1$. The estimate of the K−functional

$K\left(t, T, \Pi_{p_1,q_1;r}(E, F), \Pi_{p_2,q_2;r}(E, F)\right) =$

$$= \inf\left\{\pi_{p_1,q_1;r}(S) + t\cdot\pi_{p_2,q_2;r}(T-S) : S\in\Pi_{p_1,q_1;r}(E,F)\right\} \geq$$
$$\inf\left\{\left\|\{Sx_i\}\right\|_{p_1,q_1}^{strong} + t\cdot\left\|\{(T-S)x_i\}\right\|_{p_2,q_2}^{strong} : S\in\Pi_{p_1,q_1;r}(E,F)\right\} \geq$$
$$\inf\left\{\left\|\{y_i\}\right\|_{p_1,q_1}^{strong} + t\cdot\left\|\{Tx_i - y_i\}\right\|_{p_2,q_2}^{strong} : y_1,...,y_n\in F\right\} =$$
$$= K\left(t, \{Tx_i\}_i, l_{p_1,q_1}^{strong}(F), l_{p_2,q_2}^{strong}(F)\right)$$

implies that

$$\|T\|_{\left(\Pi_{p_1,q_1;r}(E,F),\Pi_{p_2,q_2;r}(E,F)\right)_{\theta,q}} \geq \widehat{c}\cdot\left\|\{Tx_i\}\right\|_{\left(l_{p_1,q_1}^{strong}(F),l_{p_2,q_2}^{strong}(F)\right)_{\theta,q}}.$$

But we know that $\left\|\{Tx_i\}\right\|_{\left(l_{p_1,q_1}^{strong}(F),l_{p_2,q_2}^{strong}(F)\right)_{\theta,q}} \geq \widetilde{\widetilde{c}}\cdot\left\|\{Tx_i\}\right\|_{l_{p,q}^{strong}(F)}$. Therefore, by taking the supremum over all $\{x_i\}_{i=1}^n\in\mathcal{F}(E)$ with $\|\{x_i\}\|_r^{weak} = 1$, we get that

$$\|T\|_{\left(\Pi_{p_1,q_1;r}(E,F),\Pi_{p_2,q_2;r}(E,F)\right)_{\theta,q}} \geq c\cdot\pi_{p,q;r}(T).$$

In conclusion $\left(\Pi_{p_1,q_1;r}(E,F), \Pi_{p_2,q_2;r}(E,F)\right)_{\theta,q} \subset \Pi_{p,q;r}(E,F)$, as wanted. □

We now recall some results concerning the Lorentz-Zygmund sequence spaces, which were introduced by C.Bennet and K. Rudnick, [3], and generalize the Lorentz sequence spaces.

Definition 13. ([3], [4]) Let $1\leq p,q\leq\infty$ and $-\infty<\gamma<\infty$. The Lorentz-Zygmund sequence spaces are defined as follows

$$l_{p,q,\gamma} = \left\{\xi = \{\xi_n\}_n \in c_0 : \sum_{n=1}^{\infty}\left[n^{\frac{1}{p}-\frac{1}{q}}\cdot(1+\log n)^{\gamma}\cdot s_n(\xi)\right]^q < \infty\right\},$$

if $q<\infty$, and

$$l_{p,\infty,\gamma} = \left\{\xi = \{\xi_n\}_n \in c_0 : \sup_n\left[n^{\frac{1}{p}}\cdot(1+\log n)^{\gamma}\cdot s_n(\xi)\right] < \infty\right\}.$$

Remark 1. ([4]) The formulas

$$\|\cdot\|_{p,q,\gamma} := \left(\sum_{n=1}^{\infty}\left[n^{\frac{1}{p}-\frac{1}{q}}\cdot(1+\log n)^{\gamma}\cdot s_n(\cdot)\right]^q\right)^{\frac{1}{q}},$$

respectively

$$\|\cdot\|_{p,\infty,\gamma} := \sup_n\left[n^{\frac{1}{p}}\cdot(1+\log n)^{\gamma}\cdot s_n(\cdot)\right],$$

define quasi-norms on $l_{p,q,\gamma}$, respectively on $l_{p,\infty,\gamma}$.

The **lexicografic order** of the Lorentz-Zygmund sequence spaces is important for our proofs so we establish it here.

Proposition 14. The following inclusions hold:

1. $l_{p_0,q,\gamma_0} \subseteq l_{p_1,q,\gamma_1}$, for $1 \leq p_0 < p_1 < \infty$, $1 \leq q \leq \infty$, $-\infty < \gamma_0, \gamma_1 < \infty$;
2. $l_{p,q_0,\gamma} \subseteq l_{p,q_1,\gamma}$, for $1 \leq p < \infty$, $1 \leq q_0 < q_1 \leq \infty$, $\gamma > 0$.

   Moreover, in the first case, there is a constant c_1 such that $\|x\|_{p_1,q,\gamma_1} \leq c_1 \cdot \|x\|_{p_0,q,\gamma_0}$ for every $x \in l_{p_0,q,\gamma_0}$ and in the second case there is a constant c_2 such that $\|x\|_{p,q_1,\gamma} \leq c_2 \cdot \|x\|_{p,q_0,\gamma}$ for every $x \in l_{p,q_0,\gamma}$.

To prove this proposition, we shall need the following results.

Theorem 15. ([4]) *Let $0 < q \leq \infty$ and let $\varphi, \rho \in \mathcal{B}$ $\alpha_{\overline{\rho}} < \beta_{\overline{\varphi}}$. Then $\lambda^q(\varphi)$is continuously embedded in $\lambda^q(\rho)$, where*

$$\lambda^q(\varphi) = \left\{\xi = \{\xi_n\}_n \in c_0 : \sum_{n=1}^{\infty} [\varphi(n) \cdot s_n(\xi)]^q \cdot n^{-1} < \infty\right\},$$

if $q < \infty$, and $\lambda^\infty(\varphi) = \left\{\xi = \{\xi_n\}_n \in c_0 : \sup_n [\varphi(n) \cdot s_n(\xi)] < \infty\right\}$.

In [14] N. Tita has established a relation between Lorentz spaces and Lorentz-Zygmund spaces, which is content of the next result.

Theorem 16. *Let $1 \leq p, q \leq \infty$, $0 < \gamma < \infty$ and $\xi = \{\xi_n\}_n \in c_0$. Then $\xi \in l_{p,q,\gamma} \Leftrightarrow \left\{2^{\frac{n-1}{p}} \cdot s_{2^{n-1}}(\xi)\right\}_n \in l_{r,q}$ where $\gamma = \frac{1}{r} - \frac{1}{q}$. Moreover, there are constants $\widetilde{c}(p,q,\gamma)$ and $\overline{c}(p,q,\gamma)$ such that $\widetilde{c}(p,q,\gamma) \cdot \left\|\left\{2^{\frac{n-1}{p}} \cdot s_{2^{n-1}}(\xi)\right\}_n\right\|_{r,q} \leq \|\xi\|_{p,q,\gamma} \leq$*
$$\leq \overline{c}(p,q,\gamma) \cdot \left\|\left\{2^{\frac{n-1}{p}} \cdot s_{2^{n-1}}(\xi)\right\}_n\right\|_{r,q}.$$

Proof of Proposition 14. 1. Consider $\varphi : (0,\infty) \to (0,\infty)$ defined by $\varphi(t) = t^{\frac{1}{p_0}} \cdot (1 + \log|t|)^{\gamma_0}$ and $\rho : (0,\infty) \to (0,\infty)$ defined by $\rho(t) = t^{\frac{1}{p_1}} \cdot (1 + \log|t|)^{\gamma_1}$. Then $\varphi, \rho \in \mathcal{B}$ and $\beta_{\overline{\varphi}} = \frac{1}{p_0}$, $\alpha_{\overline{\rho}} = \frac{1}{p_1}$, [4]. Hence if $0 < p_0 < p_1 < \infty$, then $\alpha_{\overline{\rho}} < \beta_{\overline{\varphi}}$, and Theorem 15 applies to give the desired inclusion.

To prove 2., note that by Theorem 16 $\xi \in l_{p,q_0,\gamma} \Leftrightarrow \left\{2^{\frac{n-1}{p}} \cdot s_{2^{n-1}}(\xi)\right\}_n \in l_{r_0,q_0}$ where $\gamma = \frac{1}{r_0} - \frac{1}{q_0}$. Let $q_1 > q_0$ and r_1 such that $\gamma = \frac{1}{r_1} - \frac{1}{q_1}$. It follows that $r_0 < r_1$ and further on $l_{r_0,q_0} \subseteq l_{r_1,q_1}$. So we obtain that $\left\{2^{\frac{n-1}{p}} \cdot s_{2^{n-1}}(\xi)\right\}_n \in l_{r_1,q_1}$ which then implies that $\xi \in l_{p,q_1,\gamma}$.

Remark 2. We must give here an explanation. In [14] there were given results for the operator ideals $L^{(s)}_{p,q,\gamma}$, where s is an additive and multiplicative s-scale, an $s-$scale being a rule $s : T \in L(E,F) \to \{s_n(T)\} \in l_\infty$ which assigns to every linear and bounded operator a bounded scalar sequence with the following properties:

1. $\|T\| = s_1(T) \geq s_2(T) \geq ... \geq 0$, for all $T \in L(E,F)$;

2. $s_{n+m-1}(T+S) \leq s_n(T) + s_m(S)$, for all $T, S \in L(E,F)$ and $n, m \in \{1, 2, ...\}$;
3. $s_{n+m-1}(T \circ S) \leq s_n(T) \cdot s_m(S)$, for all $T \in L(F, F_0), S \in L(E,F)$ and $n, m \in \{1, 2, ...\}$;
4. $s_n(T) = 0$, $\dim T < n$;
5. $s_n(I_E) = 1$, if $\dim E \geq n$, where $I_E(x) = x$, for all $x \in E$.

We call $s_n(T)$ the n-th $s-$number of the operator T. For properties, examples of $s-$numbers and relations between different s-numbers we refer the reader to [10], [12], [13].

If we take account of the similarity between the axioms of the sequence $\{s_n(T)\}_n$, where s is an additive s-scale, $T \in L(E,F)$, and the properties of $\{s_n(x)\}_n$, where $x = \{x_n\}_n \in l_\infty$, we can transfer the result obtained in [14] by N. Tita from $L^{(s)}_{p,q,\gamma}$ to $l_{p,q,\gamma}$.

In [14], an interpolation result for the Lorentz-Zygmund operator ideals $L^{(s)}_{p,q,\gamma}$ is also established. We can also transfer this to the sequence spaces case, as follows.

Theorem 17. *Let* $1 \leq p_0 < p_1 < \infty$, $1 \leq q_0 \leq q_1 \leq \infty$, $1 \leq q \leq \infty$, $0 < \gamma_0, \gamma_1 < \infty$ *and* $0 < \theta < 1$. *Then*

$$\left(l_{p_0,q_0,\gamma_0}, l_{p_1,q_1,\gamma_1}\right)_{\theta,q} \subseteq l_{p,q,\gamma},$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $\gamma = (1-\theta) \cdot \gamma_0 + \theta \cdot \gamma_1$.

Moreover for every $x \in \left(l_{p_0,q_0,\gamma_0}, l_{p_1,q_1,\gamma_1}\right)_{\theta,q}$ *the following inequality is true*

$$\|x\|_{p,q,\gamma} \leq c(p,q,\gamma) \cdot \|x\|_{\left(l_{p_0,q_0,\gamma_0}, l_{p_1,q_1,\gamma_1}\right)_{\theta,q}}.$$

We start now our construction which generalizes Miyazaki's spaces..

Definition 18. Let $1 \leq p < \infty$, $1 \leq q \leq \infty$, $-\infty < \gamma < \infty$. The vector sequence $\{x_n\}_n$ in E is strongly $(p,q,\gamma)-$summable if $\{\|x_n\|\}_n \in l_{p,q,\gamma}$. We denote by $l^{strong}_{p,q,\gamma}(E)$ the set of all such sequences in E. It is easy to see that $l^{strong}_{p,q,\gamma}(E)$ is a vector space under pointwise operations, and a natural quasi-norm is given by $\|\{x_n\}\|^{strong}_{p,q,\gamma} := \|\{\|x_n\|\}\|_{p,q,\gamma}$.

Remark 3. It is not hard to verify that all the above results for $l_{p,q,\gamma}$ can be transferred to $l^{strong}_{p,q,\gamma}(E)$.

Definition 19. Suppose that $1 \leq p,q,r \leq \infty$ and $-\infty < \gamma < \infty$. An operator $T \in L(E,F)$ is called $(p,q,\gamma;r)$ −absolutely summing provided there exists a constant $c > 0$ such that $\|\{Tx_m\}\|_{p,q,\gamma}^{strong} \leq c \cdot \|\{x_m\}\|_r^{weak}$, for every $\{x_m\}_m \in \mathcal{F}(E)$. We denote by $\Pi_{p,q,\gamma;r}(E,F)$ the space of such operators acting between E and F.

The smallest number c for which the above inequality holds is denoted by $\pi_{p,q,\gamma;r}(T)$.

Remark 4. It is routine to verify that the constant coming from $\|\cdot\|_{p,q,\gamma}^{strong}$ can be used to prove the triangle inequality, and thus $\pi_{p,q,\gamma;r}(\cdot)$ is a quasi-norm on the space of $(p,q,\gamma;r)$ −absolutely summing operators.

Theorem 20. *Let* $1 \leq p_1 < p_2 < \infty$, $1 \leq q_1 \leq q_2 \leq \infty$, $1 \leq q,r \leq \infty$, $0 < \gamma_1, \gamma_2 < \infty$ *and* $0 < \theta < 1$. *If* $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ *and* $\gamma = (1-\theta)\cdot\gamma_1 + \theta\cdot\gamma_2$ *then*

$$\left(\Pi_{p_1,q_1,\gamma_1;r}(E,F), \Pi_{p_2,q_2,\gamma_2;r}(E,F)\right)_{\theta,q} \subset \Pi_{p,q,\gamma;r}(E,F).$$

Proof. We shall use the idea from the case of $(p,q;r)$ −absolutely summing operators. First we must prove that $\left(\Pi_{p_1,q_1,\gamma_1;r}(E,F), \Pi_{p_2,q_2,\gamma_2;r}(E,F)\right)$ is an interpolation couple.

Let $T \in \Pi_{p_1,q_1,\gamma_1;r}(E,F)$ and $\{x_i\}_i \in \mathcal{F}(E)$. It follows that there exists a constant $c > 0$ such that $\|\{Tx_i\}\|_{p_1,q_1,\gamma_1}^{strong} \leq c \cdot \|\{x_i\}\|_r^{weak}$. But $\|\{Tx_n\}\|_{p_2,q_2,\gamma_2}^{strong} \leq \overline{c} \cdot \|\{Tx_n\}\|_{p_1,q_1,\gamma_1}^{strong}$. Hence we obtain

$$\|\{Tx_n\}\|_{p_2,q_2,\gamma_2}^{strong} \leq \overline{c} \cdot \|\{Tx_n\}\|_{p_1,q_1,\gamma_1}^{strong} \leq \widetilde{c} \cdot \|\{x_i\}\|_r^{weak}.$$

¿From this it follows that $T \in \Pi_{p_2,q_2,\gamma_2;r}(E,F)$, and therefore $\Pi_{p_1,q_1,\gamma_1;r}(E,F) \subset \Pi_{p_2,q_2,\gamma_2;r}(E,F)$.

Let now $T \in \Pi_{p_2,q_2,\gamma_2;r}(E,F)$ and pick $\{x_i\}_{i=1}^n \in \mathcal{F}(E)$ with $\|\{x_i\}\|_r^{weak} = 1$. The estimate of the K−functional $K\left(t, T, \Pi_{p_1,q_1,\gamma_1;r}(E,F), \Pi_{p_2,q_2,\gamma_2;r}(E,F)\right) =$
$\inf\left\{\pi_{p_1,q_1,\gamma_1;r}(S) + t\cdot\pi_{p_2,q_2,\gamma_2;r}(T-S) : S \in \Pi_{p_1,q_1,\gamma_1;r}(E,F)\right\} \geq$

$$\inf\left\{\|\{Sx_i\}\|_{p_1,q_1,\gamma_1}^{strong} + t\cdot\|\{(T-S)x_i\}\|_{p_2,q_2,\gamma_2}^{strong} : S \in \Pi_{p_1,q_1,\gamma_1;r}(E,F)\right\} \geq$$
$$\inf\left\{\|\{y_i\}\|_{p_1,q_1,\gamma_1}^{strong} + t\cdot\|\{Tx_i - y_i\}\|_{p_2,q_2,\gamma_2}^{strong} : y_1,...,y_n \in F\right\} =$$
$$= K\left(t, \{Tx_i\}_i, l_{p_1,q_1,\gamma_1}^{strong}(F), l_{p_2,q_2,\gamma_2}^{strong}(F)\right),$$ implies that

$$T_{\left(\Pi_{p_1,q_1,\gamma_1;r}(E,F),\Pi_{p_2,q_2,\gamma_2;r}(E,F)\right)_{\theta,q}} \geq \widehat{c}\cdot\|\{Tx_i\}\|_{\left(l_{p_1,q_1,\gamma_1}^{strong}(F), l_{p_2,q_2,\gamma_2}^{strong}(F)\right)_{\theta,q}}.$$

But we know that $\|\{Tx_i\}\|_{\left(l^{strong}_{p_1,q_1}(F),l^{strong}_{p_2,q_2}(F)\right)_{\theta,q}} \geq \widetilde{\widetilde{c}} \cdot \|\{Tx_i\}\|^{strong}_{p,q,\gamma}$. Taking the supremum, over all these $\{x_i\}_{i=1}^n \in \mathcal{F}(E)$, we get

$$\|T\|_{\left(\Pi_{p_1,q_1,\gamma_1;r}(E,F),\Pi_{p_2,q_2,\gamma_2;r}(E,F)\right)_{\theta,q}} \geq c \cdot \pi_{p,q;r}(T).$$

In conclusion $\left(\Pi_{p_1,q_1,\gamma_1;r}(E,F), \Pi_{p_2,q_2,\gamma_2;r}(E,F)\right)_{\theta,q} \subset \Pi_{p,q,\gamma;r}(E,F)$. □

We can further on generalize the Miyasaki operator classes. First we introduce some vector-valued sequence spaces.

Definition 21. Let $1 \leq p,q < \infty$, $-\infty < \gamma < \infty$. The vector sequence $\{x_n\}_n$ in E is weakly (p,q,γ) −summable if the scalar sequences $\{|\langle x^*, x_n\rangle|\}_n$ are in $l_{p,q,\gamma}$ for every $x^* \in E^*$. We denote by $l^{weak}_{p,q,\gamma}(E)$ the set of all such sequences in E.

Proposition 22. Suppose that $1 \leq q < p < \infty$ and $\gamma < 0$, or $1 \leq q < p < \infty$ and $0 < \gamma$ such that $\frac{1}{q} - \frac{1}{p} \geq \gamma$. Then $l^{weak}_{p,q,\gamma}(E)$ is a vector space under pointwise operations, and the formula

$$\|\{x_n\}\|^{weak}_{p,q,\gamma} := \sup_{x^* \in U_{E^*}} \left(\sum_{n=1}^{\infty}\left[n^{\frac{1}{p}-\frac{1}{q}}(1+\log n)^{\gamma}|\langle x^*, x_n\rangle|\right]^q\right)^{\frac{1}{q}}$$

defines a quasi-norm $\|\cdot\|^{weak}_{p,q,\gamma} : l^{weak}_{p,q,\gamma}(E) \to \mathbb{R}_+$.

Proof. The first step is to show that the quantity in the right side of the formula is finite. We shall apply the closed graph theorem like in the case of absolutely p−summing operators, cf. [5]. Let $x = \{x_n\}_n \in l^{weak}_{p,q,\gamma}(E)$ and associate with it the map $u : E^* \to l_{p,q,\gamma}$ given by $u(x^*) = \{\langle x^*, x_n\rangle\}_n$. Note that u is a well-defined linear map. Consider now a sequence $\{x^*_k\}_k$ which converges to x^*_0 in E^*.Then for each n, the scalar sequence $\{\langle x^*_k, x_n\rangle\}_k$ converges to $\langle x^*_0, x_n\rangle$. Thus, if we take into account the fact that $\left\{n^{\frac{1}{p}-\frac{1}{q}}(1+\log n)^{\gamma}\right\} \in c_0$, for which purpose we have made the choice of p,q and γ, we obtain as a consequence, that u has closed graph. Therefore, u is bounded. In other words

$$\|u\| = \sup_{x^* \in U_{E^*}} \left(\sum_{n=1}^{\infty}\left[n^{\frac{1}{p}-\frac{1}{q}}(1+\log n)^{\gamma}|\langle x^*, x_n\rangle|\right]^q\right)^{\frac{1}{q}} < \infty.$$

Now it is easy to check that $\|\cdot\|^{weak}_{p,q,\gamma}$ is a quasi-norm on $l^{weak}_{p,q,\gamma}(E)$. □

Definition 23. Let $1 \leq p,q < \infty$ and $-\infty < \gamma < \infty$. Suppose that $1 \leq s < r < \infty$ and $\alpha < 0$, or $1 \leq s < r < \infty$ and $0 < \alpha$ are such that $\frac{1}{s} - \frac{1}{r} \geq \alpha$. An operator

$T \in L(E,F)$ is called $(p,q,\gamma;r,s,\alpha)$ −absolutely summing if there exists a constant $c \geq 0$ such that $\|\{Tx_i\}\|_{p,q,\gamma}^{strong} \leq c \cdot \|\{x_i\}\|_{r,s,\alpha}^{weak}$ for every $\{x_i\} \in \mathcal{F}(E)$. We denote by $\Pi_{p,q,\gamma;r,s,\alpha}(E,F)$ the space of such operators acting between E and F.

The smallest number c for which the above inequality holds is denoted by $\pi_{p,q,\gamma;r,s,\alpha}(T)$.

Remark 5. It is straightforward to verify that the constant coming from $\|\cdot\|_{p,q,\gamma}^{strong}$ can be used to prove the triangle inequality and thus $\pi_{p,q,\gamma;r,s,\alpha}(\cdot)$ is a quasi-norm on the space of $(p,q,\gamma;r,s,\alpha)$ −absolutely summing operators.

Remark 6. Using the domination theorem it is routine to prove that

$$\Pi_{p,q,\gamma;p,q,\gamma}(E,F) \supseteq \Pi_q(E,F).$$

Moreover $\pi_q(T) \geq \pi_{p,q,\gamma;p,q,\gamma}(T)$ for every $T \in \Pi_q(E,F)$.

If the sequence $\alpha_n = \left\{n^{\frac{q}{p}-1} \cdot (1+\log n)^{\gamma \cdot q}\right\}$ is a decreasing one then $\Pi_{p,q,\gamma;p,q,\gamma}(E,F)$ is of the type $\Pi_{\Phi,\Psi}(E,F)$, where Φ, Ψ are symmetric norming function.

The following theorem, which is a representation result for our class of operators, will be the essential ingredient in our main theorem.

Theorem 24. *Let $1 \leq p,q < \infty$ and $-\infty < \gamma < \infty$. Suppose that $1 \leq s < r < \infty$ and $\alpha < 0$, or $1 \leq s < r < \infty$ and $0 < \alpha$ are such that $\frac{1}{s} - \frac{1}{r} \geq \alpha$. Then an operator $T \in L(E,F)$ is $(p,q,\gamma;r,s,\alpha)$ −absolutely summing if and only if $\widehat{T}\left(l_{r,s,\alpha}^{weak}(E)\right)$ is contained in $l_{p,q,\gamma}^{strong}(F)$, where $\widehat{T}:\{x_i\}_i \to \{Tx_i\}_i$. In this case*

$$\left\|\widehat{T}: l_{r,s,\alpha}^{weak}(E) \to l_{p,q,\gamma}^{strong}(F)\right\| = \pi_{p,q,\gamma;r,s,\alpha}(T).$$

The proof is similar to the case of p−absolutely summing operators, cf. [5], so we omit it.

We are now ready to state our main result.

Theorem 25. *Let $1 \leq p,q < \infty$ and $-\infty < \gamma < \infty$. Suppose that $1 \leq s < r < \infty$ and $\alpha < 0$, or $1 \leq s < r < \infty$ and $0 < \alpha$ are such that $\frac{1}{s} - \frac{1}{r} \geq \alpha$. Let also $0 < \theta < 1$. Then*

$$\left(\Pi_{p_1,q_1,\gamma_1;r,s,\alpha}(E,F), \Pi_{p_2,q_2,\gamma_2;r,s,\alpha}(E,F)\right)_{\theta,q} \subseteq \Pi_{p,q,\gamma;r,s,\alpha}(E,F),$$

where $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ *and* $\gamma = (1-\theta) \cdot \gamma_1 + \theta \cdot \gamma_2$.

Proof. We shall use an idea owed to A. Pietsch, see [10], Proposition 1.2.6. First we must prove that $\left(\Pi_{p_1,q_1,\gamma_1;r,s,\alpha}(E,F), \Pi_{p_2,q_2,\gamma_2;r,s,\alpha}(E,F)\right)$ is an interpolation couple.

Let $T \in \Pi_{p_1,q_1,\gamma_1;r,s,\alpha}(E,F)$ and $\{x_i\}_i \in \mathcal{F}(E)$. It follows that there exists a constant $\widetilde{c} > 0$ such that $\|\{Tx_i\}\|^{strong}_{p_1,q_1,\gamma_1} \leq \widetilde{c} \cdot \|\{x_i\}\|^{weak}_{r,s,\alpha}$. But we know that $\|\{Tx_n\}\|^{strong}_{p_2,q_2,\gamma_2} \leq \overline{c} \cdot \|\{Tx_n\}\|^{strong}_{p_1,q_1,\gamma_1}$. Therefore

$$\|\{Tx_i\}\|^{strong}_{p_2,q_2,\gamma_2} \leq \overline{c} \cdot \|\{Tx_i\}\|^{strong}_{p_1,q_1,\gamma_1} \leq c \cdot \|\{x_i\}\|^{weak}_{r,s,\alpha}.$$

In conclusion $T \in \Pi_{p_2,q_2;r,s,\alpha}(E,F)$ and $\Pi_{p_1,q_1,\gamma_1;r,s,\alpha}(E,F) \subset \Pi_{p_2,q_2,\gamma_2;r,s,\alpha}(E,F)$.

Let now $\{x_i\}_i \in l^{weak}_{r,s,\alpha}(E)$. We define the operator $X : T \in L(E,F) \to \{Tx_i\}_i$. It follows from the preceding representation theorem that $\{Tx_i\}_i \in l^{strong}_{p_1,q_1,\gamma_1}(F)$, if $T \in \Pi_{p_1,q_1,\gamma_1;r,s,\alpha}(E,F)$ and $\{Tx_i\}_i \in l^{strong}_{p_2,q_2,\gamma_2}(F)$, if $T \in \Pi_{p_2,q_2,\gamma_2;r,s,\alpha}(E,F)$. Thus

$$\begin{aligned} X &: \Pi_{p_1,q_1,\gamma_1;r,s,\alpha}(E,F) \to l^{strong}_{p_1,q_1,\gamma_1}(F), \\ X &: \Pi_{p_2,q_2,\gamma_2;r,s,\alpha}(E,F) \to l^{strong}_{p_2,q_2,\gamma_2}(F), \end{aligned}$$

are linear and bounded.

It now follows from the interpolation Theorems 6 and 8 that

$$X : \left(\Pi_{p_1,q_1,\gamma_1;r,s,\alpha}(E,F), \Pi_{p_2,q_2,\gamma_2;r,s,\alpha}(E,F)\right)_{\theta,q} \to$$

$$\left(l^{strong}_{p_1,q_1,\gamma_1}(E), l^{strong}_{p_2,q_2,\gamma_2}(E)\right)_{\theta,q} \subseteq l^{strong}_{p,q,\gamma}(E)$$

Hence the assertion follows from the representation theorem. □

Acknowledgements. I would like to express my gratitude to N. Tita for many fruitfull discussions.

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