STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 4, December 2001

SOME DUALITY THEOREMS FOR LINEAR-FRACTIONAL PROGRAMMING HAVING THE COEFFICIENTS IN A SUBFIELD K OF REAL NUMBERS

DOINA IONAC

Abstract. In this paper we obtain some duality results for linear-fractional programming having the coefficients in a field K of real numbers, having the property that the rational numbers set $\mathbb{Q} \subseteq K$.

1. We consider the following linear-fractional programming problem:

(PF). Find

$$Max \left\{ f(x) \equiv \frac{cx + c_0}{dx + d_0} | Ax \leq b, x \geq 0 \right\},$$

where $A = (a_{ij})$ is a $m \times n$ matrix $(m < n)$ with $rankA = m$ and with the elements in K, c and d are n-vectors with components in K, b is a m-vector with components in K and c_0 and d_0 are constants in K.

Let X be the feasible set in $\mathbb{R}^n$ of the problem PF, and let XK be the feasible set of the problem PF in K^n defined by $XK = X \cap K^n$.

Next, we suppose that:

(a) $dx + d_0 > 0$, $\forall x \in X$,

and also we will denote by E the set

$$E = \{x \in \mathbb{R}^n | x \geq 0, dx + d_0 > 0\}.$$

Obviously, $X \subseteq E$ and E is a convex set.

Definition 1. i) The problem (PF) is called **regular** if X is nonempty, f is not constant and it exists $M > 0$ such that: $0 < dx + d_0 < M$, $\forall x \in X$.

ii) The problem (PF) is called **pseudo-regular** if X is non-empty, f is not constant and verifies the condition (a).

Key words and phrases. Fractional programming, duality, regular and pseudo-regular programs.

Next we study some duality properties for problem (PF) under the hypothesis of regularity and pseudo-regularity which is larger than considered, for instance, by Shesham [5]. Some examples [2'] show that there are regular linear-fractional problems with unbounded feasible solution set X having an infinite optimum value.

2. A way to construct a dual for problem (PF) is by transforming it into a linear programming problem by the variable change $y = tx$:

(PL). Find

$$Max\{(cy + c_0 t | Ay - bt \leq 0, dy + d_0 t = 1, y \geq 0, t \geq 0\}.$$

Let the objective function of problem (PL) be denoted by $g(y,t) = cy + c_0 t$.

Also let XL be the feasible set of the problem (PL).

The following property extends to the case when (PF) is regular a similar result obtained by Charnes-Cooper [1] (see, [3], [6], [7]) under the supposition that X is a bounded nonempty set.

Theorem 2. *(i) If the problem (PF) is regular then for every feasible solution (y,t) of the problem (PL), we have $t > 0$.*

(ii) If problem (PF) satisfy the condition (a), then for any $x' \in X$, there exists $(y',t') \in XL$, such that $x' = \frac{y'}{t'}$ and $f(x') = g(y',t')$.

(iii) If problem (PF) is regular then for any feasible solution (y',t') of the problem (PL) there exists a feasible solution $x' \in X$ of (PF) such that $x' = \frac{y'}{t'}$ and $f(x') = g(y',t')$.

Next we need an auxiliary results which establishes the relationship between problems (PF) and (PL) that generalizes for regular linear-fractional programming a result obtained by Charnes-Cooper [1] (see, also [3], [6]) in the case when the feasible set is bounded and nonempty:

Theorem 3. *If the problem (PF) is regular, then only one of the following statements holds:*

(i) The problems (PF) and (PL) have both optimal solutions and its optimal values are equal and finite. Moreover, if (y^*,t^*) is an optimal solution of (PL) then $x^* = \frac{y^*}{t^*}$ is an optimal solution for (PF) and conversely if x'' is an optimal solution of (PF) there exists an optimal solution (y'',t'') of (PL) such that $y'' = t''x''$.

(ii) The problems (PF) and (PL) have both infinite optimal values, that is $\sup\{f(x)|x \in X\} = \sup\{g(y,t)|(y,t) \in XL\} = +\infty$.

3. Let XK be the feasible set of the problem (PFK),

(PFK). Find

$$Max\left\{\frac{cx+c_0}{dx+d_0}|Ax \leq b,\, x \geq 0,\, x \in K^n\right\},$$

that is $XK = X \cap K^n$.

We consider also the auxiliary linear programming problem (PLK) on the field K associated to the problem (PL):

(PLK). Find

$$Max\{(cy+c_0t)|Ay-bt \leq 0, dy+d_0t = 1, y \geq 0, t \geq 0, (y,t) \in K^{n+1}\}.$$

Remark 1. The feasible set XK is a polytop (i.e. the intersection of a finite number of closed semi-spaces) and since $a_{ij},\ c_j,\ b_i \in K$, $(i = 1,2,...,m$ and $j = 1,2,...,n)$, any extremal points of X is an element in K^n.

The following result establishes a relationship between (PF) and (PFK):

Theorem 4. *If the condition (a) holds then the following statements are true:*

(i) Problem (PF) is infeasible (i.e. $X = \emptyset$*) if and only if the problem (PFK) is infeasible (i.e.* $XK = \emptyset$*).*

(ii) Problem (PF) has no optimal solution, if and only if (PFK) has no optimal solution.

(iii) Problem (PF) has an optimal solution if and only if the problem (PFK) has an optimal solution.

(iv) If x' *is an optimal solution of the problem (PF) and* x'' *is an optimal solution for problem (PFK), then* $f(x') = f(x'')$.

Proof. The proof of this theorem has the same main idea that a similar result for rational programming problems from [4]. □

4. We can associate to problem (PF) a linear dual (see, [6]) which is the dual of the auxiliary linear programming problem (PL):

DL. Find

$$Min\ z,$$

subject to:

$A^t u + dz \geq c,$

$bu - d_0 z \leq -c_0,$

$u \geq 0,\ u \in \mathbb{R}^m,\ z \in \mathbb{R}.$

Let consider the problem:

DLK. Find

$$Min\ z,$$

subject to

$A^t u + dz \geq c,$

$bu - d_0 z \leq -c_0$, and

$u \geq 0,\ z \in K,\ u \in K^m.$

Theorem 5. *If the problem (PF) is regular only one of the statement holds:*

(i) Both problems (PF) and (DL) (primal and dual) has feasible solutions. In this case, both problems have optimal solutions and its optimal values are equal.

(ii) problem (PF) has feasible solutions and (DL) has not feasible solutions. In this case, the problem (PF) has an infinite optimum.

Proof. Since by hypothesis (PF) is regular, it has feasible solutions, and by Theorem 2, problem (PL) has feasible solutions too. Then by linear programming duality theorem only one of the following two cases holds:

a) the dual problem (DL) is a feasible problem;

b) the dual (DL) is an unfeasible problem.

(i) If the dual problem (DL) is a feasible problem then the optimal values of the primal and dual problems are equal. But then, by Theorem 3, the optimal values of the problems (PF) and (PL) are equal.

(ii) If the dual problem (DL) is an unfeasible problem then, by the linear programming duality theorem, it follows that problem (PL) has an infinite optimal value. But then, by Theorem 3, the problem (PF) has an infinite optimal value too. □

Theorem 6. *If the problem (PFK) is regular only one of the following statements holds:*

(i) Both primal and dual problems (PFK) and (DLK) have feasible solutions. In this case, both problems have optimal solutions and its optimal values are equal.

(ii) Problem (PFK) has feasible solutions and (DLK) has not. In this case, the problem (PFK) has an infinite optimum.

Proof. By duality Theorem 5, only one of the following cases holds:

a) Both primal and dual problems (PF) and (DL) are feasible problems. In this case, both problems have optimal solutions and its optimal values are equal.

b) Problem (PF) is a feasible problem and (DL) is an unfeasible problem. In this case the problem (PF) has an infinite optimum.

(i) Let consider the case when both problems (PF) and (DL) have feasible solutions. Then, by Theorem 4 and by Theorem 3.2 [2], it follows that the problems (PFK) and (DLK) have optimal solutions and its optimal values are equal with that of (PF) and (DL), respectively. Therefore, under the theorem hypotheses the problems (PFK) and (DLK) have optimal solutions and its optimal values are equal.

(ii) On the other part, when (DL) is unfeasible, from Theorem 5 it follows that problem (PF) has an infinite optimum. But then by Theorem 3.2 [2], (DLK) is an unfeasible problem and by Theorem 4 (PFK) has an infinite optimum. □

References

[1] Charnes A., Cooper W. W., Programming with linear fractional functionals, Naval Res. Logist. Quart., 9 (1962) 1-2, 181-186.
[2] Ionac D., Some duality theorems for linear programming with the coefficients in a field K of real numbers, A.C.A.M., Tehnical University of Cluj-Napoca 6, 2 (1997), 50-54.
[2'] Ionac D., Metode de rezolvare a problemelor de programare matematică, Teză de doctorat, Univ. Babeş-Bolyai, Cluj-Napoca, 1998.
[3] Lasdon L. S., Optimization Theory of Large Systems, Macmillan Co., New York, 1970.
[4] Lupsa L., Duality Theorems for rational programming problems, Analyse Numer. Theor. Approx., 16 (1987) 1, 51-53.
[5] Sesham C. R., On duality in linear fractional programming, Proc. Indian Acad. Sci., Sect. A, Math. Sci., 62 (6) (1980), 119-121.
[6] Stancu-Minasian I. M., Metode de rezolvare a problemelor de programare fracţionară, Editura Academiei Române, Bucureşti, 1992.
[7] Stancu-Minasian I. M., Tigan S., On some methods for solving fractional programming problems with inexact data, Stud. cerc. Mat., 45 (1993), no. 6, 517-532.

University of Oradea, Str. Armatei Române 5, 3700 Oradea, Romania