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A GENERALIZED INVERSION FORMULA AND SOME APPLICATIONS

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Abstract. In this paper we shall establish a general result involving Dirichlet product of arithmetical functions, which provides information on the subtle properties of the integers.

1. **Introduction and preliminaries**

The Möbius function $\mu(n)$ is defined as follows:

$$\mu(1) = 1, \quad \mu(q_1 \cdot q_2 \cdot \dots \cdot q_k) = (-1)^k$$

if all the primes $q_1, q_2, \dots, q_k$ are different; $\mu(n) = 0$ if n has a squared factor. The Möbius inversion formula is a remarkable tool in numerous problems involving integers and there are other inversion formulas involving $\mu(n)$. In particular, we obtain the following well-known theorem:

If

$$G(x) = \sum_{n=1}^{\lfloor x \rfloor} F\left(\frac{x}{n}\right)$$

for all positive x, $(x \geq 1)$, then

$$F(x) = \sum_{n=1}^{\lfloor x \rfloor} \mu(n) G\left(\frac{x}{n}\right)$$

and conversely.

Many of these inversion formulas can be written in the form of a single formula which generalizes them all.

2. **The main result**

First of all, we establish the following theorem:

Theorem 1. *Given arithmetical functions* $\alpha, \beta, u : \mathbb{N}^* \to \mathbb{C}$ *such that*

$$\alpha * \beta = u, \quad u(n) = \begin{cases} 1, & \text{if} \quad n = 1 \\ 0, & \text{if} \quad n \geq 2 \end{cases}$$

let $h : A \times \mathbb{N}^* \to A$ *be a function, such that:*

a) $h(x, 1) = x$ *for all* $x \in A$*, where* $A \subset \mathbb{C}$*,* $A \neq \emptyset$*.*

b) $h(h(x, n), k)$ *is constant for* x *constant and* $nk = constant$*, where* $x \in A$ *and* $n, k \in \mathbb{N}^*$*.*

Let $F, G : \mathbb{C} \to \mathbb{C}$ *be functions such that* $F(x) = G(x) = 0$ *for all* $x \in \mathbb{C} \setminus A$*. Suppose that the both series:*

$$\sum_{n,k \in \mathbb{N}^*} \alpha(n)\beta(k)G(h(h(x, n), k)), \qquad \sum_{n,k \in \mathbb{N}^*} \beta(n)\alpha(k)F(h(h(x, n), k))$$

converge absolutely.

Then, for all $x \in A$*, we have*

$$F(x) = \sum_{n \in \mathbb{N}^*} \beta(n)G(h(x, n)) \tag{1}$$

if and only if

$$G(x) = \sum_{n \in \mathbb{N}^*} \alpha(n)F(h(x, n)). \tag{2}$$

Proof. Suppose that (1) is true. It follows that

$$\sum_{n \in \mathbb{N}^*} \alpha(n)F(h(x, n)) = \sum_{n \in \mathbb{N}^*} \alpha(n) \sum_{k \in \mathbb{N}^*} \beta(k)G(h(h(x, n), k)) =$$

$$= \sum_{n \in \mathbb{N}^*} \sum_{k \in \mathbb{N}^*} \alpha(n)\beta(k)G(h(h(x, n), k)).$$

An absolutely convergent series can be rearranged in an arbitrary way without affecting the sum. We have

$$\alpha(1)\beta(1) = 1, \quad G(h(h(x, 1), 1)) = G(h(x, 1)) = G(x).$$

We can arrange the terms as follows:

$$\sum_{\substack{n,k,d \in \mathbb{N}^* \\ nk = d \neq 1}} \alpha(n)\beta(k)G(h(h(x, n), k)) = \sum_{d \in \mathbb{N}^*, d \neq 1} G(h(h(x, n), k)) \sum_{\substack{n,k \in \mathbb{N}^* \\ nk = d, d \neq 1}} \alpha(n)\beta(k) = 0,$$

because $\alpha * \beta = u$.

Therefore

$$\sum_{n\in\mathbb{N}^*}\alpha(n)F(h(x,n))=G(x).$$

Conversely, (2) implies (1) and hence the theorem is proved.

3. **Examples**

1) Letting $A=[0,\infty)$, $h: A\times\mathbb{N}^*\to A$,

$$h(x,n)=\frac{x}{n},\quad h(x,1)=\frac{x}{1}=x \text{ for all } x\in A;$$

$$h(h(x,n),k)=h\left(\frac{x}{n},k\right)=\frac{x}{nk}=\text{constant}$$

for $nk=$ constant and $x=$ constant.

Consider the mappings $F,G:[0,\infty)\to\mathbb{C}$ such that $F(x)=G(x)=0$ for all $x\in[0,1)$. We deduce from theorem 1, that

$$F(x)=\sum_{n\le x}\beta(n)G\left(\frac{x}{n}\right)$$

and

$$G(x)=\sum_{n\le x}\alpha(n)F\left(\frac{x}{n}\right)$$

are equivalent. Moreover, if we let $\alpha(n)=1$ for all $n\in\mathbb{N}^*$, we deduce that

$$F(x)=\sum_{n\le x}\mu(n)G\left(\frac{x}{n}\right)$$

and

$$G(x)=\sum_{n\le x}F\left(\frac{x}{n}\right)$$

are equivalent for all positive x, $(x\ge 1)$.

2) Let us denote by $\overline{P}(x)$ the number of the integers $k\in\mathbb{N}^*$ such that $k\le x$, $k\ne a^b$ for all $a,b\in\mathbb{N}^*$, $b\ge 2$. It is known that

$$\sum_{2^n\le x}\overline{P}(x^{1/n})=\lfloor x-1\rfloor.$$

We deduce from theorem 1 that

$$\sum_{2^n\le x}\mu(n)\lfloor x^{1/n}-1\rfloor=\overline{P}(x).$$

3) The number $Q(x)$ of squarefree numbers not exceeding x satisfies

$$\sum_{x/n^2 \geq 1} Q(x/n^2) = \lfloor x \rfloor.$$

If we use theorem 1, we have

$$\sum_{x/n^2 \geq 1} \mu(n) \left\lfloor \frac{x}{n^2} \right\rfloor = Q(x).$$

4) If $|z| < 1$, we have

$$\frac{z}{1-z} = \sum_{n \in \mathbb{N}^*} z^n.$$

Letting $A = U(0,1)$, $h(z,n) = z^n$, $F(z) = z$, $G(z) = \dfrac{z}{1-z}$, $\alpha(n) = 1$, $\beta(n) = \mu(n)$ for all $n \in \mathbb{N}^*$, we have:

$$\sum_{n,k \in \mathbb{N}^*} \beta(n)\alpha(k)F(h(h(z,n),k)) = \sum_{n,k \in \mathbb{N}^*} \mu(n) z^{nk}$$

$$\sum_{\substack{n,k,d \in \mathbb{N}^* \\ nk=d=const}} |\mu(n) z^{nk}| \leq \sum_{\substack{n,k,d \in \mathbb{N}^* \\ nk=d=const}} |z^{nk}| = \sum_{d \in \mathbb{N}^*} \sum_{nk=d} |z|^{nk} \leq \sum_{d \in \mathbb{N}^*} d|z|^d$$

(because $\sum_{nk=d} |z|^{nk} \leq d|z|^d$).

It is possible to apply Cauchy's test:

$$\lim_{d \to \infty} \sqrt[d]{d|z|^d} = \lim_{d \to \infty} \sqrt[d]{d} \cdot |z| = |z| < 1.$$

It follows that series $\sum_{n,k \in \mathbb{N}^*} \beta(n)\alpha(k)F(h(h(z,n),k))$ converges absolutely for all $z \in U(0,1)$. We can also show that $\sum_{n,k \in \mathbb{N}^*} \alpha(n)\beta(k)G(h(h(z,n),k))$ converges absolutely. We deduce from theorem 1 that

$$\frac{z}{1-z} = \sum_{n \in \mathbb{N}^*} z^n$$

and

$$z = \sum_{n \in \mathbb{N}^*} \mu(n) \frac{z^n}{1-z^n},$$

are equivalent for all $z \in U(0,1)$.

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