STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 4, December 2001

SOME HOMEOMORPHISM THEOREMS

FLORICA ALDEA

Abstract. In this paper we give homeomorfism result for operators that satisfies Borsuk condition.

1. **Introduction**

Let X be a Banach space and $f : X \to X$ be an operator such that $F_f \neq \emptyset$. There are many papers in which using the fixed point theory we obtain the surjectivity of $\mathbf{1}_X - f$ (see: Aldea [1, 2], Browder [4], Danes [8], Danes-Kolomy [9], Deimling [10], Rus [14, 15, 16].

The aim of this paper is to give an answer to the following question. What conditions must satisfy f such that $\mathbf{1}_X - f$ be a homeomorphism?

Rus proved in [15] that if f is a φ contraction then $\mathbf{1}_X - f$ is a homeomorphism. In order to prove this he used a bijectivity and a data dependence results.

Also, it is possible to obtain homeomorphism result using domain invariance result respective closing range theorem (see: Cramer-Ray [6], Crandall-Pazzy [7], Dowing-Kirk [11], Zeidler [17]).

Following a similar technique we will give an answer to the mention question in case that operator f satisfy Borsuk condition.

Definition 1.1. Let X be a Banach space and $f : X \to X$ an operator. We say that f satisfies Borsuk condition (shortly (B)), if there exists $\eta > 0$ and $\varepsilon > 0$ such that for all $x_1,\ x_2 \in X$, inequality

$$||f(x_1) - f(x_2)|| < \eta$$

1991 *Mathematics Subject Classification.* 47H10.

Key words and phrases. homeomorphism, fixed points, Borsuk condition.

implies

$$||x_1 - x_2|| < \varepsilon.$$

Now we will give some operators' classes which satisfy (B) condition.

Remark 1.1. Let X be a Banach space. If $f : X \to X$ is near identity (in Campanato sense [5]), then f satisfies condition (B).

Proof. Because f is near $\mathbf{1}_X$ there exists constants $\lambda,\ k \in (0, 1)$ such that

$$||x_1 - x_2 - \lambda(f(x_1) - f(x_2))|| \leq k \cdot ||x_1 - x_2||, \text{ for all } x_1,\ x_2 \in X \tag{1}$$

or

$$(1 - k)||x_1 - x_2|| \leq \lambda||f(x_1) - f(x_2)||, \text{ for all } x_1,\ x_2 \in X.$$

So there are $\eta > 0$ and $\varepsilon \left(= \dfrac{\lambda}{1-k}\eta\right) > 0$ such that from $||f(x_1) - f(x_2)|| < \eta$ we have $||x_1 - x_2|| < \varepsilon$. We obtain that f verifies condition (B).

Remark 1.2. Let X be Banach space. If $f : X \to X$ is dilatation, then f satisfies (B) condition.

Proof. Because f is dilatation there exists $c > 1$ such that

$$c||x_1 - x_2|| \leq ||f(x_1) - f(x_2)||, \text{ for all } x_1,\ x_2 \in X$$

So there are $\eta > 0$ and $\varepsilon \left(= \dfrac{\eta}{c}\right) > 0$ such that from $||f(x_1) - f(x_2)|| < \eta$ we have $||x_1 - x_2|| < \varepsilon$. We obtain that f verifies condition (B).

Remark 1.3. Let X Banach space. If $f : X \to X$ is strong accretive, then f satisfies condition (B).

Proof. Because f is strong accretive there is $k > 1$ such that

$$k||x_1 - x_2|| \leq ||f(x_1) - f(x_2)||, \text{ for all } x_1,\ x_2 \in X$$

So there are $\eta > 0$ and $\varepsilon \left(= \dfrac{\eta}{k}\right) > 0$ such that from $||f(x_1) - f(x_2)|| < \eta$ we have $||x_1 - x_2|| < \varepsilon$. We obtain that f verifies (B) condition.

Definition 1.2. (Rus, [15]) A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function if φ is increasing and $\varphi^n(t) \to 0$ when $n \to \infty$ for all $t \in \mathbb{R}_+$.

2. **Main result**

In what follows, we solve the problem for case of an operator which is sum of two operators and one of them satisfies condition (B).

Theorem 2.1. (Granas, [12]) *Let X be a Banach space and operator $F : X \to X$ be a complete continuous . If operator $f : X \to X$ satisfies condition (B) (with $f(x) = x - F(x)$ for all $x \in X$), then f is surjective.*

Theorem 2.2. *Let X be a Banach space, $F,\ L : X \to X$ be two continuous operators with F compact and functions $\varphi,\ \psi : [0,\infty) \to [0,\infty)$. Suppose that:*

(i)

$$\varphi(||x_1 - x_2||) \leq ||f(x_1) - f(x_2)|| \tag{2}$$

for all $x_1,\ x_2 \in X$ with $f(x) = \mathbf{1}_X(x) - F(x)$, for all $x \in X$;

(ii)

$$||L(x_1) - L(x_2)|| \leq \psi(||x_1 - x_2||) \tag{3}$$

for all $x_1,\ x_2 \in X$;

(iii) $\varphi(0) = 0,\ \varphi$ bijective and φ^{-1} comparison function;

(iv) $\psi(0) = 0$ and ψ comparison function.

Then $\mathbf{1}_X - f$ is bijective.

Proof. First, we prove that $F_{F+L} = \emptyset$. In order to apply Theorem 2.1 we will prove that f verifies condition (B). Let $x_1,\ x_2$ from X such that $||f(x_1)-f(x_2)|| < \eta$. From (2) and φ bijective we have

$$\begin{aligned} \varphi(||x_1 - x_2||) &\leq ||f(x_1) - f(x_2)|| < \eta \\ ||x_1 - x_2|| &\leq \varphi^{-1}(\eta) < \varphi^{-1}(\eta) + 1 = \varepsilon \end{aligned}$$

so f verifies condition (B).

From Theorem 2.1 we have that f is surjective. From (2) and (iii) we obtain that f is injective. Operator f is continuous from hypothesis and continuity of inverse operator results from inequality (2); so f is homeomorphism.

Let $x \in X$, because f is homeomorphism we define operator

$$R : X \to X;\ x \longmapsto R(x)$$

such that

$$f(R(x)) = L(x) \text{ for all } x \in X.$$

From (2) and (3) we have that

$$\begin{aligned}\varphi(||R(x_1) - R(x_2)||) &\leq ||f(R(x_1)) - f(R(x_2))|| = ||L(x_1) - L(x_2)|| \\ &\leq \psi(||x_1 - x_2||)\end{aligned}$$

for all $x_1,\ x_2 \in X$. Because φ is invertible

$$||R(x_1) - R(x_2)|| \leq (\varphi^{-1} \circ \psi)(||x_1 - x_2||) \tag{4}$$

for all $x_1,\ x_2 \in X$.

Because $\varphi^{-1},\ \psi$ are comparison functions we obtain that

$$||R(x_1) - R(x_2)|| \leq \varphi^{-1}(||x_1 - x_2||) \tag{5}$$

for all $x_1,\ x_2 \in X$. But φ^{-1} is comparison function. From the last statement and (4) we apply fixed point theorem for φ-contractions (see Rus [16]) we have $F_R = \{x^*\}$. From the definition of R results

$$(\mathbf{1}_X - F)(x^*) = L(x^*) \iff F_{F+L} = \{x^*\}.$$

Second, we prove that $\mathbf{1}_X - (F + L)$ is bijective.

Let $y \in X$. We denote by L_y operator $L+y$. It is easy to prove that operator L_y verifies inequality (3), so applying first part of our proof we have that $F_{F+L_y} = \{x^*\} \iff$ equation $F(x) + L(x) + y = x$ has only one solution. So $\mathbf{1}_X - (F + L)$ is bijective.

Theorem 2.3. *If we add to the hypotheses of Theorem 2.2 the following:*

(v) $\varphi(t) \geq \psi(t)$ *for all* $t \geq 0$;

(vi) there is the inverse of $\mathbf{1}_{[0,\infty)} - (\varphi^{-1} \circ \psi)$ *and it is continuous.*

Then $\mathbf{1}_X - (F + L)$ *is homeomorphism.*

Proof. From Theorem 2.2 we have that operator $\mathbf{1}_X - (F+G)$ is bijective, continuity of its results from the continuity of F and L.

Let x_i the unique solution of equations $x - F(x) - L(x) = y_i$, for $i = 1,\ 2$.

From (2) and (3) we have

$$\begin{aligned}\varphi(||x_1-x_2||) &\leq ||f(x_1)-f(x_2)|| = ||L(x_1)-L(x_2)+y_1-y_2|| \\ &\leq ||L(x_1)-L(x_2)|| + ||y_1-y_2|| \\ &\leq \psi(||x_1-x_2||) + ||y_1-y_2|| \Longrightarrow\end{aligned}$$

From (iii) resuts

$$\begin{aligned}||x_1-x_2|| &\leq (\varphi^{-1}\circ\psi)(||x_1-x_2||) + \varphi^{-1}(||y_1-y_2||) \\ &\leq (\varphi^{-1}\circ\psi)(||x_1-x_2||) + ||y_1-y_2||) \iff \\ (\mathbf{1}_{[0,\infty)} - (\varphi^{-1}\circ\psi))(||x_1-x_2||) &\leq ||y_1-y_2|| \Longrightarrow \\ ||x_1-x_2|| &\leq (\mathbf{1}_{[0,\infty)} - (\varphi^{-1}\circ\psi))^{-1}(||y_1-y_2||) \qquad (6)\end{aligned}$$

From last inequality and (vi) we have that

$$||(\mathbf{1}_X-(F+L))^{-1}(y_1)-(\mathbf{1}_X-(F+L))^{-1}(y_2)|| \leq (\mathbf{1}_{[0,\infty)}-(\varphi^{-1}\circ\psi))^{-1}(||y_1-y_2||)$$

Which means that $(\mathbf{1}_X-(F+L))^{-1}$ is continuous operator, so $\mathbf{1}_X-(F+L)$ homeomorphism.

Remark 2.1. If X is finite dimensional Banach space, then Theorems 2.1, 2.2 are true without assumption of compactness on operator F.

Theorem 2.4. (Altman, [3]) *Let X be a finite dimensional Banach space, $F,\ L : X\to X$ two continuous operators and constants $c>0$ and $k>0$. Suppose that:*

(i)

$$c\cdot||x_1-x_2|| \leq ||f(x_1)-f(x_2)|| \qquad (7)$$

for all $x_1,\ x_2\in X$ with $f(x)=\mathbf{1}_X(x)-F(x)$, for all $x\in X$;

(ii)

$$||L(x_1)-L(x_2)|| \leq k\cdot||x_1-x_2|| \qquad (8)$$

for all $x_1,\ x_2\in X$;

(iii)

$$K<c.$$

Then

(a) $F_{F+L}=\{x^*\}$;

(b) $\mathbf{1}_X - (F+L) : X \to X$ *is homeomorphism;*

(c) Operator $(\mathbf{1}_X - (F+L))^{-1} : X \to X$ *is Lipschitz continuous.*

Proof. In order to prove theorem, we apply Theorem 2.2 and 2.3 considering $\varphi(t) = c \cdot t$ with $c > 1$ and $\psi(t) = k \cdot t$ with $k < 1$.

These functions verify assumption (i)-(v) from mentioned theorems.

Function $(\mathbf{1}_{[0,\infty)} - (\varphi^{-1} \circ \psi))(t) = \dfrac{c-k}{c} t$ verifies (vi).

Conclusion (c) of Altman's theorem results from inequality (6).

References

[1] F. Aldea, *Surjectivity via fixed point structures,* Seminar on Fixed Point Theory "Babeş-Bolyai" Univ., 1997, 9-14.

[2] F. Aldea, *Surjectivity theorems for normcontraction operators,* to appear in Mathematica.

[3] M. Altman, *Dilating mappings, implicit functions and fixed point theorems in finite-dimensional,* Fundamenta Matematicae, LXVIII(1970), 129-141.

[4] F.Browder, *Fixed Point Theory and Nonlinear Problems,* Bull. Amer. Math. Soc., 9(1983), 1-39.

[5] S. Campanato, *A Cordes type condition for nonlinear and variational systems,* Rend. Acad. Naz. Sci., 107(1989), 307-321.

[6] W.J. Cramer, W.O. Ray, *Solvability of nonlinear operator equations,* Pacific J. Math., 95(1981), 37-50.

[7] M. Crandall, A. Pazzy, *On the range of accretive operators,* Israel J. Math., 27(1977), 235-246.

[8] J. Danes, *Surjectivity and fixed point theorems,* C. M. U. C., 13(1972), 181-184.

[9] J. Danes, J. Kolomy, *Fixed points, surjectivity and invariance of domain theorems for weakly continuous mappings,* Boll. U.M.I., 5 (1976), 369-394.

[10] K.Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, 1985.

[11] D. Dowing, W. Kirk, *A generalization of Caristi theorem with application to nonlinear mappings theory,* Pacific J. Math., 69(1977), 339-349.

[12] A. Granas, *Uber ein Satz von K. Borsuk,* Bull. Acad. Polon. Sci., 5(1957), 595-562.

[13] S. Kasahara, *Surjectivity and fixed point of nonlinear mappings*, Math. Sem. Notes, 2 (1974), 119-126.

[14] I. A. Rus, *Generalized contraction,* Seminar on fixed point theory, "Babeş-Bolyai" Univ., 1983, 1-131.

[15] I. A. Rus, *Ecuaţii diferenţiale, ecuaţii integrale şi sisteme dinamice,* Transilvania Press, 1996.

[16] I. A. Rus, *Contracţii generalizate,* to appear.

[17] E. Zeidler, *Nonlinear functional analysis and this applications. I. Fixed point theorems,* Springer, 1990.

Babeş-Bolyai University, Department of Applied Mathematics,
str. M. Kogalniceanu 1, 3400 Cluj-Napoca, Romania
E-mail address: aldea@math.ubbcluj.ro